Continuity, Derivatives and Differentiability

Let $f : \mathbb{R}^2 \longmapsto \mathbb{R}$.

Def.: 1°. Let $(x_0, y_0) \in \mathbb{R}^2$. f is continuous at (x_0, y_0) if $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

2°.The partial derivatives of f with respect to x and y are defined by

$$f_x(x,y) \equiv \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \quad \text{and} \quad f_y(x,y) \equiv \lim_{\Delta y\to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

respectively if the limits exist.

3°. Let $\overrightarrow{u} = \langle a, b\rangle$ be a unit vector. The directional derivative of f in the direction $\overrightarrow{u}$ is defined by

$$D_{\overrightarrow{u}} f(x,y) = \lim_{t\to 0} \frac{f(x+ta, y+tb) - f(x,y)}{t}$$

if the limit exists.

4°. f is differentiable at (x,y) if $\exists \varepsilon_1, \varepsilon_2$ satisfying $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$ s.t.

$$f(x+\Delta x, y+\Delta y) = f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y.$$

Examples:

1. Continuous $\not\Rightarrow f_x, f_y$ exist.

   Let $f(x,y) = |x| \Rightarrow f$ is continuous at $(0,0)$ but $f_x(0,0)$ does not exist.

2. f_x, f_y exist $\not\Rightarrow$ Continuous.

   Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0) \end{cases}$.

   $\therefore f_x(0,0) = \displaystyle\lim_{\Delta x\to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{0-0}{\Delta x} = 0$ and, similarly, $f_y(0,0) = 0$.

   Along the line $y = mx$:

   $\because f(x,y) = f(x, mx) = \dfrac{mx^2}{(1+m^2)x^2} = \dfrac{m}{1+m^2} \to \dfrac{m}{1+m^2}$ as $(x,y) \to (0,0)$.

   $\therefore \displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist $\Rightarrow f$ is not continuous at $(0,0)$.

3. f_x, f_y exist $\not\Rightarrow D_{\overrightarrow{u}} f(x,y)$ exists for all unit vector $\overrightarrow{u}$.

   Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0) \end{cases}$.

   $\therefore f_x(0,0) = f_y(0,0) = 0$.

   Let $\overrightarrow{u} = \langle a, b\rangle$ with $\|\overrightarrow{u}\| = 1$ and $ab \neq 0$.

   $\therefore D_{\overrightarrow{u}} f(0,0) = \displaystyle\lim_{t\to 0} \frac{f(ta, tb) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^2 ab}{t^2(a^2+b^2)} - 0}{t} = \lim_{t\to 0} \frac{ab}{t}$ does not exist.

4. f_x, f_y exist $\not\Rightarrow$ Differentiable.

   Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0) \end{cases}$.

   $\therefore f_x(0,0) = f_y(0,0) = 0$ and f is not continuous at $(0,0) \Rightarrow f$ is not differentiable at $(0,0)$.

5. $D_{\overrightarrow{u}} f(x,y)$ exists for all unit vector $\overrightarrow{u} \not\Rightarrow$ Differentiable.

   Let $f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0) \end{cases}$.

   Let $\overrightarrow{u} = \langle a, b \rangle$ with $\|\overrightarrow{u}\| = 1$.

   $\therefore D_{\overrightarrow{u}} f(0,0) = \lim\limits_{t\to 0} \dfrac{f(ta,tb) - f(0,0)}{t} = \lim\limits_{t\to 0} \dfrac{\frac{t^3a^3}{t^2(a^2+b^2)} - 0}{t} = a^3 \Rightarrow f_x(0,0) = 1 \;\&\; f_y(0,0) = 0.$

   Assume that f is differentiable at $(0,0) \Rightarrow a^3 = D_{\overrightarrow{u}} f(0,0) = f_x(0,0)a + f_y(0,0)b = a. \quad \rightarrow\leftarrow.$

   (Note that f is continuous at $(0,0)$.)

6. $D_{\overrightarrow{u}} f(x,y)$ exists for all unit vector $\overrightarrow{u} \not\Rightarrow$ Continuous.

   Let $f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0) \end{cases}$.

   Let $\overrightarrow{u} = \langle a, b \rangle$ with $\|\overrightarrow{u}\| = 1$.

   $\therefore D_{\overrightarrow{u}} f(0,0) = \lim\limits_{t\to 0} \dfrac{f(ta,tb) - f(0,0)}{t} = \lim\limits_{t\to 0} \dfrac{\frac{t^3a^2b}{t^2(t^2a^4+b^2)} - 0}{t} = \begin{cases} \dfrac{a^2}{b} & , \text{ if } b \neq 0 \\ 0 & , \text{ if } b = 0 \end{cases}$.

   $\because \lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist $\Rightarrow f$ is not continuous.

7. Differentiable $\Rightarrow$ Continuity and $f_x, f_y, D_{\overrightarrow{u}} f$ exist.

8. f_x, f_y exist and continuous $\Rightarrow$ Differentiable.

9. Differentiable $\not\Rightarrow$ f_x, f_y exist and continuous.

   Let $f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0) \end{cases}$.

   $\therefore f_x(0,0) = \lim\limits_{\Delta x\to 0} \dfrac{(\Delta x)^2 \sin\dfrac{1}{|\Delta x|} - 0}{\Delta x} = \lim\limits_{\Delta x\to 0} \Delta x \cdot \sin\dfrac{1}{|\Delta x|} = 0$ and $f_y(0,0) = 0$.

   and for $(x,y) \neq (0,0)$,

$$\frac{\partial f}{\partial x}(x,y) = 2x\sin\frac{1}{\sqrt{x^2+y^2}} - \frac{x\cos\dfrac{1}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y}(x,y) = 2y\sin\frac{1}{\sqrt{x^2+y^2}} - \frac{y\cos\dfrac{1}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}.$$

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{\partial f}{\partial x}(x,y)$ and $\lim\limits_{(x,y)\to(0,0)} \dfrac{\partial f}{\partial y}(x,y)$ do not exist.

$\Big(\because \dfrac{\partial f}{\partial x}(x,0) = 2x\sin\dfrac{1}{|x|} - \dfrac{x}{|x|}\cos\dfrac{1}{|x|}.$

$\therefore \dfrac{\partial f}{\partial x}(x,y)$ oscillates and does not exist when the limit is taken along x-axis.$\Big)$

$\therefore \dfrac{\partial f}{\partial x}(x,y)$ and $\dfrac{\partial f}{\partial x}(x,y)$ are not continuous at $(0,0)$.

Let $\varepsilon_1 = \Delta x \sin\dfrac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$ and $\varepsilon_2 = \Delta y \sin\dfrac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$.

$\Rightarrow \varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$.

$\because f(0,0) = f_x(0,0) = f_y(0,0) = 0$

$$\begin{aligned}\Rightarrow f(\Delta x, \Delta y) &= \big((\Delta x)^2 + (\Delta y)^2\big)\sin\frac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}} \\ &= f(0,0) + f_x(0,0)\Delta x + f_y(0,0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y.\end{aligned}$$

$\therefore f$ is differentiable at $(0,0)$.