

CONTENTS

10 Conics, Parametric Equations, and Polar Coordinates	1
10.1 Summary	1
Index	17

LIST OF TABLES

LIST OF FIGURES

Chapter 10

CONICS, PARAMETRIC EQUATIONS, AND POLAR COORDINATES

10.1 Summary

Section 10.1 Conics and calculus 3

1. Three ways to define the conics

(a) the intersections of planes and cones

(b) general second-degree equation (一般二次方程式)

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

(c) **locus** (軌跡) (collection) of points satisfying a certain geometric property

.....7

2. **Standard equation of a parabola**

The **standard form**

(標準式) of the equation of a parabola with vertex (h, k) and directrix $y = k - p$ is

$$(x - h)^2 = 4p(y - k). \quad \text{Vertical axis}$$

For directrix $x = h - p$, the equation is

$$(y - k)^2 = 4p(x - h). \quad \text{Horizontal axis}$$

The focus lies on the axis p units (*directed distance*) from the vertex.

The coordinates of the focus are as follows.

$(h, k + p)$ Vertical axis $(h + p, k)$ Horizontal axis

..... 10

3. **Reflective property of a parabola** Let P be a point on a parabola.

The tangent line to the parabola at the point P makes equal angles with the following two lines.

(a) The line passing through P and the focus.

(b) The line passing through P parallel to the axis of the parabola.

..... 17

4. **Standard equation of an ellipse** The standard form of the equation of an ellipse with center (h, k) and major and minor axes of

lengths $2a$ and $2b$, where $a > b$, is

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \quad \text{Major axis is horizontal}$$

or

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1. \quad \text{Major axis is vertical}$$

The foci lie on the major axis, c units from the center, with $c^2 = a^2 - b^2$.

21

5. **Reflecting property of an ellipse** Let P be a point on an ellipse.

The tangent line to the ellipse at point P makes equal angles with the lines through P and the foci. 26

6. **Eccentricity of an ellipse** The eccentricity (離心率) e of an

ellipse is given by the ratio

$$e = \frac{c}{a}.$$

..... 28

7. **Standard equation of a hyperbola** The standard form of the equation of a hyperbola with center at (h, k) is

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \quad \text{Transverse axis is horizontal}$$

or

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1. \quad \text{Transverse axis is vertical}$$

The vertices are a units from the center, and the foci are c units from the center, where $c^2 = a^2 + b^2$ 38

8. **Asymptotes of a hyperbola** For a horizontal transverse axis

(水平貫軸), the equation of the asymptotes are

$$y = k + \frac{b}{a}(x - h) \quad \text{and} \quad y = k - \frac{b}{a}(x - h).$$

For a vertical transverse axis (垂直貫軸), the equation of the asymptotes are

$$y = k + \frac{a}{b}(x - h) \quad \text{and} \quad y = k - \frac{a}{b}(x - h).$$

..... 41

9. **Eccentricity of a hyperbola** The eccentricity (離心率) e of a hyperbola is given by the ratio

$$e = \frac{c}{a}.$$

..... 44

Section 10.2 Plane curves and parametric equations 50

10. **Plane curve** If f and g are continuous functions of t on an interval I , then the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are called **parametric equations** (參數方程式) and t is called the **parameter** (參數). The set of points (x, y) obtained as t varies over the interval I is called the **graph** (圖) of the parametric equations. Taken together, the parametric equations and the graph are called a **plane curve** (平面曲線), denoted by C 54

11. **Smooth curve** (平滑曲線) A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called **smooth** (平滑) if f' and g' are continuous on I and not simultaneously 0, except possibly at the

endpoints of I . The curve C is called **piecewise smooth** (片段平滑) if it is smooth on each subinterval of some partition of I 71

Section 10.3 Parametric equations and calculus 77

12. **Parametric form of the derivative** If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at (x, y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0.$$

. 80

13. If $dy/dt = 0$ and $dx/dt \neq 0$ when $t = t_0$, the curve represented by $x = f(t)$ and $y = g(t)$ has a horizontal tangent at $(f(t_0), g(t_0))$. Similarly, if $dx/dt = 0$ and $dy/dt \neq 0$ when $t = t_0$, the curve represented by $x = f(t)$ and $y = g(t)$ has a vertical tangent at $(f(t_0), g(t_0))$ 88

14. **Arc length in parametric form** If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \leq t \leq b$ (except possibly at the endpoints), then the arc length of C over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.$$

..... 89

15. **Area of a surface of revolution** If a smooth curve C given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \leq t \leq b$, then the area S of the surface of revolution formed by revolving C about the coordinate axes is given by the following.

$$1. S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Revolution about the x -axis

$$2. S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Revolution about the y -axis

..... 98

Section 10.4 Polar coordinates and polar graphs 102

16. **Polar-to-rectangular conversion** The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of the point as follows.

$$1. x = r \cos \theta \text{ and } y = r \sin \theta. \quad 2. \tan \theta = \frac{y}{x} \text{ and } r^2 = x^2 + y^2. \dots 109$$

17. **Slope in polar form** If f is a differentiable function of θ , then the

slope of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}$$

provided that $dx/d\theta \neq 0$ at (r, θ) 120

18. **Tangent lines at the pole** If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the line $\theta = \alpha$ is tangent at the pole to the graph of $r = f(\theta)$ 125

Section 10.5 Area and arc length in polar coordinates 131

19. **Area in polar coordinates** If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

given by

$$A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta, \quad 0 < \beta - \alpha \leq 2\pi.$$

..... 135

20. **Arc length of a polar curve** Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

..... 149

21. **Area of a surface of revolution** Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The area of the

surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line as follows.

$$(a) S = 2\pi \int_{\alpha}^{\beta} y \, ds = 2\pi \int_{\alpha}^{\beta} f(\theta) \sin \theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} \, d\theta \quad \text{About the } y\text{-axis}$$

$$(b) S = 2\pi \int_{\alpha}^{\beta} x \, ds = 2\pi \int_{\alpha}^{\beta} f(\theta) \cos \theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} \, d\theta \quad \text{About the } x\text{-axis}$$

..... 153

Section 10.6 Polar equation of conics and Kepler's law 156

22. Classification of conics by eccentricity (用離心率對圓錐曲線分類)

Let F be a fixed point (focus) and let D be a fixed line (directrix) in the plane. Let P be another point in the plane and let e (eccentricity (離心率)) be the ratio of the distance between P and F to the distance between P and D . The collection of all points P with a given eccentricity is a conic.

(a) The conic is an ellipse if $0 < e < 1$.

(b) The conic is a parabola if $e = 1$.

(c) The conic is a hyperbola if $e > 1$.

..... 157

23. **Polar equations of conics** The graph of a polar equation of the form

$$r = \frac{ed}{1 \pm e \cos \theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e \sin \theta}$$

is a conic, where $e > 0$ is the eccentricity and $|d|$ is the distance between the focus at the pole and its corresponding directrix. 162

24. The four types of equations indicated in Theorem 10.17 can be classified as follows, where $d > 0$.

a. Horizontal directrix above the pole: $r = \frac{ed}{1 + e \sin \theta}$

b. Horizontal directrix below the pole: $r = \frac{ed}{1 - e \sin \theta}$

c. Vertical directrix to the right of the pole: $r = \frac{ed}{1 + e \cos \theta}$

d. Vertical directrix to the left of the pole: $r = \frac{ed}{1 - e \cos \theta}$

..... 165

25. Ellipse: $b^2 = a^2(1 - e^2)$ Hyperbola: $b^2 = a^2(e^2 - 1)$ 169

26. **Kepler's Laws** (**克卜勒定律**)

(a) Each planet moves in an elliptical orbit with the sun as a focus.

(b) A ray from the sun to the planet sweeps out equal areas of the ellipse in equal times.

(c) The square of the period is proportional to the cube of the mean distance between the planet and the sun.

..... 172

INDEX

arc length 弧長

in parametric form 參數形式, 9

of a polar curve 極座標曲線, 12

area 面積

in polar coordinates 極座標, 11

of a surface of revolution 旋轉曲面

in parametric form 參數型式, 9

in polar coordinates 極座標, 12

asymptote(s) 漸近

of a hyperbola 雙曲線, 5

axis 軸

of a parabola 拋物線, 2

Classification of conics by eccentricity 用離心率對圓錐曲線分類, 13

classification of conics by eccentricity 用離心率對圓錐曲線分類, 13

- conic 圓錐曲線 標, 10
- classification by eccentricity 離心率分類, 13
- directrix of 準線, 13
- eccentricity 離心率, 13
- focus of 焦點, 13
- parabola 拋物線, 2
- polar equations of 極座標方程式, 14
- coordinate conversion 坐標轉換
- polar to rectangular 極座標到矩形, 10
- rectangular to polar 長方形到極座標, 10
- coordinates, polar 坐標, 極
- area in 面積, 11
- area of a surface of revolution in 旋轉曲面面積, 12
- converting to rectangular 極坐標到矩形, 10
- coordinates, rectangular, converting to polar 轉換到極坐標, 矩形, 10
- curve 曲線
- piecewise smooth 分段平滑, 7
- plane 平面, 7
- smooth 平滑, 7

- piecewise 分段, 7
- derivative(s) 導數
 - parametric form 參數形式, 8
- directrix 準線
 - of a conic 圓錐曲線, 13
 - of a parabola 拋物線, 2
- eccentricity 離心率, 13
 - classification of conics by 圓錐曲線分類, 13
 - of a hyperbola 雙曲線, 6
 - of an ellipse 橢圓, 4
- ellipse 橢圓
 - eccentricity of 離心率, 4
- reflective property of 反射性, 4
- standard equation of 標準方程式, 3
- equation(s) 方程式
 - general second-degree 一般二次, 1
 - of a hyperbola 雙曲線, 5
 - of a parabola 拋物線, 2
 - of an ellipse 橢圓, 3
 - of conics, polar 圓錐曲線, 極座標, 14
 - parametric 參數, 7
 - graph of 圖, 7
- famous curves 著名曲線

- parabola 拋物線, 2
- focus 焦點
 - of a conic 圓錐曲線, 13
 - of a parabola 拋物線, 2
- general second-degree equation 一般二次方程式, 1
- graph(s) 圖
 - of parametric equations 參數方程式, 7
- hyperbola 雙曲線
 - asymptotes of 漸近線, 5
 - eccentricity of 離心率, 6
 - standard equation of 標準方程式, 5
- Kepler's Laws 克卜勒定律, 15
- length 長度
 - of an arc 弧長
 - parametric form 參數形式, 9
 - polar form 極式, 12
- line(s) 直線
 - tangent 切線
 - at the pole 極, 11
- locus 軌跡, 2
- parabola 拋物線, 2
 - axis of 軸, 2

- directrix of 準線, 2
- focus of 焦點, 2
- reflective property of 反射性, 3
- standard equation of 標準方程式, 2
- vertex of 頂點, 2
- parameter 參數, 7
- parametric equations 參數方程式, 7
- graph of 圖, 7
- parametric form 參數形式
- of arc length 弧長, 9
- of the area of a surface of revolution 旋轉體面積, 9
- of the derivative 導數, 8
- piecewise smooth curve 分段平滑的曲線, 7
- plane curve 平面曲線, 7
- polar coordinates 極座標
- area in 面積, 11
- area of a surface of revolution in 旋轉曲面面積, 12
- converting to rectangular 極坐標到矩形, 10
- polar curve, arc length of 極坐標曲線, 弧形長度, 12
- polar equations of conics 圓錐曲線

- 極座標方程式, 14
- polar form of slope 極座標形式的斜率, 10
- pole 極
- tangent lines at 切線, 11
- rectangular coordinates 直角坐標
- converting to polar 轉換到極坐標, 10
- reflective property 反射性
- of a parabola 拋物線, 3
- of an ellipse 橢圓, 4
- revolution 旋轉
- surface of 表面積
- area of 面積, 9, 12
- second-degree equation, general 二次方程式, 一般, 1
- slope(s) 斜率
- of a tangent line 切線
- parametric form 參數形式, 8
- polar form 極式, 10
- smooth 平滑
- curve 曲線, 7
- piecewise 分段, 7
- standard equation of 標準方程式
- a hyperbola 雙曲線, 5
- a parabola 拋物線, 2

- an ellipse 橢圓, 3
- standard form of the equation of 方程式的標準形式
- a hyperbola 雙曲線, 5
- a parabola 拋物線, 2
- an ellipse 橢圓, 3
- surface of revolution 旋轉體表面積
- area of 面積
 - parametric form 參數形式, 9
 - polar form 極式, 12
- tangent line(s) 切線
 - at the pole 極, 11
 - slope of 斜率
 - parametric form 參數形式, 8
 - polar form 極式, 10
- vertex 頂點
 - of a parabola 拋物線, 2
- 分段平滑的曲線 piecewise smooth curve, 7
- 切線 tangent line(s)
 - 斜率 slope of
 - 參數形式 parametric form, 8
 - 極式 polar form, 10
 - 極 at the pole, 11
- 反射性 reflective property
 - 拋物線 of a parabola, 3

- 橢圓 of an ellipse, 4
- 方程式 equation(s)
- 一般二次 general second-degree, 1
- 拋物線 of a parabola, 2
- 參數 parametric, 7
- 圖 graph of, 7
- 圓錐曲線，極座標 of conics, polar, 14
- 橢圓 of an ellipse, 3
- 雙曲線 of a hyperbola, 5
- 方程式的標準形式 standard form of the equation of
- 拋物線 a parabola, 2
- 橢圓 an ellipse, 3
- 雙曲線 a hyperbola, 5
- 一般二次方程式 general second-degree equation, 1
- 二次方程式，一般 second-degree equation, general, 1
- 平面曲線 plane curve, 7
- 平滑 smooth
- 曲線 curve, 7
- 分段 piecewise, 7
- 用離心率對圓錐曲線分類 Classification of conics by eccentricity, 13

- 用離心率對圓錐曲線分類 classification of conics by eccentricity, 13
- 曲線 curve
- 分段平滑 piecewise smooth, 7
- 平面 plane, 7
- 平滑 smooth, 7
- 分段 piecewise, 7
- 坐標，極 coordinates, polar
- 面積 area in, 11
- 旋轉曲面面積 area of a surface of revolution in, 12
- 極坐標到矩形 converting to rectangular, 10
- 坐標轉換 coordinate conversion
- 長方形到極座標 rectangular to polar, 10
- 極座標到矩形 polar to rectangular, 10
- 克卜勒定律 Kepler's Laws, 15
- 弧長 arc length
- 參數形式 in parametric form, 9
- 極座標曲線 of a polar curve, 12
- 拋物線 parabola, 2
- 反射性 reflective property of, 3
- 頂點 vertex of, 2
- 焦點 focus of, 2

- 軸 axis of, 2
- 準線 directrix of, 2
- 標準方程式 standard equation of, 2
- 面積 area
- 旋轉曲面 of a surface of revolution
- 直坐標 rectangular coordinates
- 參數型式 in parametric form, 9
- 轉換到極坐標 converting to polar, 10
- 極座標 in polar coordinates, 12
- 極座標 in polar coordinates, 11
- 直線 line(s)
- 參數 parameter, 7
- 切線 tangent
- 參數方程式 parametric equations, 7
- 極 at the pole, 11
- 圖 graph of, 7
- 長度 length
- 參數形式 parametric form
- 弧長 of an arc
- 弧長 of arc length, 9
- 參數形式 parametric form, 9
- 旋轉體面積 of the area of a surface

- of revolution, 9
- 導數 of the derivative, 8
- 斜率 slope(s)
- 切線 of a tangent line
- 參數形式 parametric form, 8
- 極式 polar form, 10
- 旋轉 revolution
- 表面積 surface of
 - 面積 area of, 9, 12
- 旋轉體表面積 surface of revolution
- 面積 area of
 - 參數形式 parametric form, 9
 - 極式 polar form, 12
- 頂點 vertex
 - 拋物線 of a parabola, 2
- 著名曲線 famous curves
 - 拋物線 parabola, 2
- 焦點 focus
 - 拋物線 of a parabola, 2
 - 圓錐曲線 of a conic, 13
- 圓錐曲線 conic
 - 拋物線 parabola, 2
 - 焦點 focus of, 13
 - 極座標方程式 polar equations of, 14
 - 準線 directrix of, 13

- 離心率 eccentricity, 13
- 離心率分類 classification by eccentricity, 13
- 圓錐曲線極座標方程式 polar equations of conics, 14
- 軸 axis
- 拋物線 of a parabola, 2
- 極 pole
- 切線 tangent lines at, 11
- 極座標 polar coordinates
- 面積 area in, 11
- 旋轉曲面面積 area of a surface of revolution in, 12
- 極坐標到矩形 converting to rectangular, 10
- 極座標曲線, 弧形長度 polar curve, arc length of, 12
- 極座標形式的斜率 polar form of slope, 10
- 準線 directrix
- 拋物線 of a parabola, 2
- 圓錐曲線 of a conic, 13
- 圖 graph(s)
- 參數方程式 of parametric equations, 7
- 漸近 asymptote(s)

- 雙曲線 of a hyperbola, 5
- 標準方程式 standard equation of
 - 拋物線 a parabola, 2
 - 橢圓 an ellipse, 3
 - 雙曲線 a hyperbola, 5
- 導數 derivative(s)
 - 參數形式 parametric form, 8
- 橢圓 ellipse
 - 反射性 reflective property of, 4
 - 標準方程式 standard equation of, 3
 - 離心率 eccentricity of, 4
- 轉換到極坐標，矩形 coordinates,
 - rectangular, converting to polar, 10
- 離心率 eccentricity, 13
- 圓錐曲線分類 classification of conics by, 13
- 橢圓 of an ellipse, 4
- 雙曲線 of a hyperbola, 6
- 雙曲線 hyperbola
 - 漸近線 asymptotes of, 5
 - 標準方程式 standard equation of, 5
 - 離心率 eccentricity of, 6