Calculus (II), Midterm Exam 2022/04/22

1. (16%) Determine whether the series converges absolutely or conditionally, or diverges. In addition, please indicate the test you use.

   (a) $\sum_{n=1}^{\infty}(-1)^n tan\frac{1}{n}$

   (b) $\sum_{n=1}^{\infty}(-\frac{1}{3})^n(1+\frac{1}{n})^{n^2}$

   (c) $\sum_{n=1}^{\infty}\frac{(-2)^n}{3\cdot5\cdot7\ldots(2n+1)}$

   (d) $\sum_{n=1}^{\infty}\frac{\ln(n+2)}{n+2}$

Ans:

(a) $\lim_{n\to\infty}\frac{tan\frac{1}{n}}{\frac{1}{n}}=1,$ since $\frac{1}{n}$ is a p-series with $p\le 1$ which is divergent. Therefore, by the limit comparison test, $\sum_{n=1}^{\infty} tan\frac{1}{n}$ diverges. In addition, since $\lim_{n\to\infty} tan\frac{1}{n}=0$ and $tan\frac{1}{n}$ is decreasing. Therefore, by the alternating series test $\sum_{n=1}^{\infty}(-1)^n tan\frac{1}{n}$ converges. All in all, $\sum_{n=1}^{\infty}(-1)^n tan\frac{1}{n}$ is conditionally converges.

(b) Since $\lim_{n\to\infty}\sqrt[n]{|a_n|}=\frac{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}{3}=\frac{e}{3}<1$. By the root test, it is absolute converges.

(c) Since $\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|=\lim_{n\to\infty}|\frac{(-2)^{n+1}}{3\cdot5\cdot7\ldots(2n+1)(2n+3)}\cdot\frac{3\cdot5\cdot7\ldots(2n+1)}{(-2)^n}|=\lim_{n\to\infty}\frac{2}{2n+3}=0.$

Therefore, by the ratio test, it is absolute converges

(d) Let $f(x)=\frac{\ln(x+2)}{x+2}$, $f'(x)=\frac{1-\ln(x+2)}{(x+2)^2}<0$ for $x\ge 1$. f is positive, continuous and decreasing for $x\ge 1$

$$\int_1^{\infty}\frac{\ln(x+2)}{x+2}dx=\lim_{b\to\infty}\frac{[\ln(x+2)]^2}{2}\Big|_1^b=\infty$$

So the series diverges by the integral test.

2. (12%) Find the interval of convergence of the power series (Be sure to check the for the convergence at the endpoints of the intervals)

(a) $\sum_{n=0}^{\infty} n!\,(x-2)^n$

(b) $\sum_{n=0}^{\infty} \frac{(x-3)^n}{3^n}$

Ans:

(a) $\lim_{n\to\infty} |\frac{u_{n+1}}{u_n}| = \lim_{n\to\infty} |\frac{(n+1)!(x-2)^{n+1}}{n!(x-2)^n}| = \infty$ which implies that the series converges only at the center 2.

(b) $\lim_{n\to\infty} |\frac{u_{n+1}}{u_n}| = \lim_{n\to\infty} |\frac{(x-3)^{n+1}/3^{n+1}}{(x-3)^n/3^n}| = \lim_{n\to\infty} |\frac{x-3}{3}|$. By the ratio test, the series converges for $\left|\frac{x-3}{3}\right| < 1 \to 0 < x < 6$

Note that when $x = 0$ $\sum_{n=0}^{\infty} \frac{(x-3)^n}{3^n} = \sum_{n=0}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n$ which is divergent by the n-th term test.

when $x = 6$ $\sum_{n=0}^{\infty} \frac{(x-3)^n}{3^n} = \sum_{n=0}^{\infty} \frac{(3)^n}{3^n} = \sum_{n=0}^{\infty} 1$ which is divergent by the n-th term test. Therefore, the interval of convergence is (0,6).

3. (12%) Let $F(x) = \int_0^x \ln(1+\frac{t^2}{2})dt$

(a) Find the Maclaurin series for $F(x)$ and its radius of convergence.

(b) Estimate $F(0.1)$ with an error less than 10^{-4}

Ans:

(a) By the fundamental theorem of calculus, $F'(x) = \ln(1+\frac{x^2}{2})$. Notice that

$$\ln(1+\frac{x^2}{2}) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\frac{x^2}{2})^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x)^{2n}}{2^n n}$$

Term by term integration yields $F(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n+1}}{2^n n(2n+1)}$

Using the ratio test $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(-1)^n x^{2n+3}}{2^{n+1}(n+1)(2n+3)} \cdot \frac{2^n n(2n+1)}{(-1)^{n-1} x^{2n+1}}| = |\frac{x^2}{2}|$.

Therefore, when $\left|\frac{x^2}{2}\right| < 1 \to |x| < \sqrt{2}$ it is convergent. The radius of convergence is $\sqrt{2}$

(b) Let $b_n = \frac{(\frac{1}{10})^{2n+1}}{2^n n(2n+1)}$

$$b_1 = \frac{\left(\frac{1}{10}\right)^3}{2^1 1(2+1)} = \frac{1}{6000} = 0.000\ \ 167 > 10^{-4}$$

$$b_2 = \frac{\left(\frac{1}{10}\right)^5}{2^2 2(4+1)} = \frac{1}{4000000} < 10^{-4}$$

Since it is an alternating series, therefore $\mathrm{F}(0.1) \sim \frac{1}{6000} = 0.00167$

4. (12%) Evaluate the following expression (Try to use the Basic series of Taylor series and notice that the power series is a continuous function)

   (a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

   (b) $\frac{\pi}{3} - \frac{\pi^3}{3^3\times 3!} + \frac{\pi^5}{3^5\times 5!} - \frac{\pi^7}{3^7\times 7!} + \ldots$

   (c) $\lim_{x\to 0}\frac{\sin(x) - x + \frac{1}{6}x^3}{x^5}$

Ans:

(a) Since $\ln x = (\mathrm{x}-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots, 0 < \mathrm{x} \le 2$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln 2$$

(b) $\frac{\pi}{3} - \frac{\pi^3}{3^3\times 3!} + \frac{\pi^5}{3^5\times 5!} - \frac{\pi^7}{3^7\times 7!} + \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{(\frac{\pi}{3})^{2n+1}}{(2n+1)!} = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

(c) $\lim_{x\to 0}\frac{\sin(x) - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to 0}\frac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots) - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to 0}\frac{(\frac{x^5}{5!} - \frac{x^7}{7!} + \cdots)}{x^5} = \frac{1}{5!} = \frac{1}{120}$

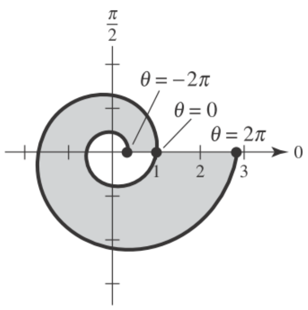
5. (10%) Find the first two nonzero terms of Taylor series of $f(x) = \csc(x)$ center at $\frac{\pi}{2}$

Ans:

$$f(x) = \csc(x)$$
$$f'(x) = -\csc(x)\cot(x)$$
$$f''(x) = \csc^3(x) + \csc(x)\,cot^2(x)$$

$$\csc(x) = 1 + \frac{1}{2!}(x - \frac{\pi}{2})^2 + \cdots$$

6. (10%) The following figure shows the polar graph of $r = e^{\frac{\theta}{6}}$ where $-2\pi \le \theta \le 2\pi$. Find the area of the shaded region



Ans:

$$A = \frac{1}{2}\int_0^{2\pi} (e^{\frac{\theta}{6}})^2\, d\theta - \frac{1}{2}\int_{-2\pi}^{0} \left(e^{\frac{\theta}{6}}\right)^2 d\theta = \frac{1}{2}\int_0^{2\pi} e^{\frac{\theta}{3}}\, d\theta - \frac{1}{2}\int_{-2\pi}^{0} e^{\frac{\theta}{3}}\, d\theta$$
$$= \frac{3}{2}e^{\frac{\theta}{3}}\Big|_0^{2\pi} - \frac{3}{2}e^{\frac{\theta}{3}}\Big|_{-2\pi}^{0} = \frac{3}{2}[e^{\frac{2\pi}{3}} + e^{\frac{-2\pi}{3}} - 2]$$

7. (10%) Find the area of the surface formed by revolving the polar graph $r = 2\sin(\theta)$ about the polar axis over the interval $0 \le \theta \le \pi$

Ans:

$$r = 2\sin(\theta)$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{4sin^2\theta + 4cos^2\theta} = 2$$

$$S = 2\pi\int_0^{\pi} 2\sin(\theta)\, sin\theta 2 d\theta = 8\pi\int_0^{\pi} \frac{1 - \cos(2\theta)}{2} = 4\pi\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\pi}$$
$$= 4\pi^2$$

8. (9%) Classify the following surface, if it is quadratic surface you should further classify it into six basic types of surface

(a) $16x^2 - y^2 + 16z^2 = 4$

(b) $\mathrm{r} = r^2 sin^2(\theta)$ (this representation is in cylindrical coordinates)

(c) $\rho = 4csc(\Phi)sec(\theta)$ (this representation is in spherical coordinates)

Ans:

(a) $16x^2 - y^2 + 16z^2 = 4 \rightarrow \frac{x^2}{(\frac{1}{2})^2} - \frac{y^2}{(2)^2} + \frac{z^2}{(\frac{1}{2})^2} = 1$

It is Hyperboloid of one sheet

(b) $\mathrm{r} = r^2 sin^2(\theta) \rightarrow r = 0 \;\; or \;\; r = csc^2\theta$ which is not a graph that we have cover in the class (此題送分)

(c) $\rho = 4cs\,c(\Phi)\,sec(\theta) = \frac{4}{\sin(\Phi)\cos(\theta)} \rightarrow x = \rho\sin(\Phi)\cos(\theta) = 4$ which is a plane

9. (9%) Evaluate the following expression

(a) $\lim_{t\to 0} \sqrt{t+1}\boldsymbol{i} + (3t+2)\boldsymbol{j} + \frac{1-\cos(t)}{t}\boldsymbol{k}$

(b) Let $\mathbf{r}(\mathrm{t}) = \sin(\mathrm{t})\,\mathbf{i} + \cos(\mathrm{t})\,\mathbf{j} + \mathrm{t}\boldsymbol{k}$, $\mathbf{u}(\mathrm{t}) = \sin(\mathrm{t})\mathbf{i} + \cos(\mathrm{t})\mathbf{j} + \frac{1}{t}\boldsymbol{k}$, find

$\frac{d}{dt}[\boldsymbol{r}(t) \cdot \boldsymbol{u}(t)]$ (Note that $\cdot$ denotes inner-product)

(c) $\int 6\boldsymbol{i} - 2t\boldsymbol{j} + \ln(t)\,\boldsymbol{k}\;\; dt$

Ans:

(a) $\lim_{t\to 0} \sqrt{t+1}\boldsymbol{i} + (3t+2)\boldsymbol{j} + \frac{1-\cos(t)}{t}\boldsymbol{k} = \boldsymbol{i} + 2\boldsymbol{j}$ since $\lim_{t\to 0} \frac{1-\cos(t)}{t} =$

$\lim_{t\to 0} \frac{\sin(t)}{1} = 0$

(b) $\boldsymbol{r}(t) \cdot \boldsymbol{u}(t) = 1 + 1 = 2$, $\frac{d}{dt}[\boldsymbol{r}(t) \cdot \boldsymbol{u}(t)] = 0$

(c) $\int 6\boldsymbol{i} - 2t\boldsymbol{j} + \ln(t)\,\boldsymbol{k}\;\; dt = 6\mathrm{t}\boldsymbol{i} - \mathrm{t}^2\boldsymbol{j} + (\mathrm{t}\ln t - \mathrm{t})\boldsymbol{k} + \boldsymbol{C}$