Chapter 9 Infinite Series

Szu-Chi Chung

Department of Applied Mathematics, National Sun Yat-sen University

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Sequences

- A sequence is defined as a function whose domain is the set of positive integers.
- Although a sequence is a function, it is common to represent sequences by subscript notation rather than by the standard function notation.
- For instance, in the sequence

$$\begin{array}{cccccccl} 1, & 2, & 3, & 4, & \ldots, & n, & \ldots & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \text{Sequence} \\ a_1, & a_2, & a_3, & a_4, & \ldots, & a_n, & \ldots & \end{array}$$

  1 is mapped onto a_1, 2 is mapped onto a_2, and so on.
- The numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ are the The number a_n is the nth term of the sequence, and the entire sequence is denoted by $\{a_n\}$.

Example 1 (Listing the terms of a sequence)

a. The terms of the sequence $\{a_n\} = \{3 + (-1)^n\}$ are

$$3 + (-1)^1,\ 3 + (-1)^2,\ 3 + (-1)^3,\ 3 + (-1)^4,\ \ldots \Longrightarrow 2,\ 4,\ 2,\ 4,\ \ldots$$

b. The terms of the sequence $\{b_n\} = \left\{\frac{n}{1-2n}\right\}$ are

$$\frac{1}{1-2\cdot 1},\ \frac{2}{1-2\cdot 2},\ \frac{3}{1-2\cdot 3},\ \frac{4}{1-2\cdot 4},\ \ldots \Longrightarrow -1,\ -\frac{2}{3},\ -\frac{3}{5},\ -\frac{4}{7},\ \ldots$$

c. The terms of the sequence $\{c_n\} = \left\{\frac{n^2}{2^n-1}\right\}$ are

$$\frac{1^2}{2^1-1},\ \frac{2^2}{2^2-1},\ \frac{3^2}{2^3-1},\ \frac{4^2}{2^4-1},\ \ldots \Longrightarrow \frac{1}{1},\ \frac{4}{3},\ \frac{9}{7},\ \frac{16}{15},\ \ldots$$

d. The terms of the recursively defined sequence $\{d_n\}$, where $d_1 = 25$ and $d_{n+1} = d_n - 5$, are

$$25,\quad 25-5=20,\quad 20-5=15,\quad 15-5=10,\quad \ldots \blacksquare$$

Limit of a sequence

- Sequences whose terms approach to limiting values, are said to For instance, the sequence $\{1/2^n\}$

$$\frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \quad \frac{1}{16}, \quad \frac{1}{32}, \quad \dots$$

converges to 0, as indicated in the following definition.

Definition 9.1 (The limit of a sequence)

Let L be a real number. The limit of a sequence $\{a_n\}$ is L, written as

$$\lim_{n\to\infty} a_n = L$$

if for each $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$. If the limit L of a sequence exists, then the sequence converges to L. If the limit of a sequence does not exist, then the sequence diverges.

- Graphically, this definition says that eventually (for $n > M$ and $\varepsilon > 0$) the terms of a sequence that converges to L will lie within the band between the lines $y = L + \varepsilon$ and $y = L - \varepsilon$ as shown in Figure 1.

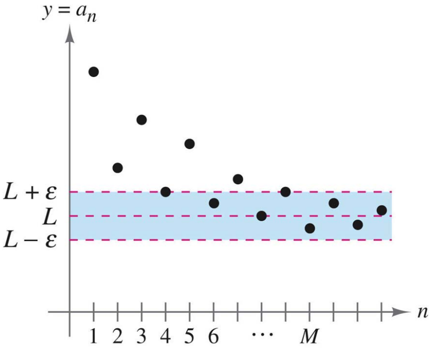


Figure 1: For $n > M$, the terms of the sequence all lie within ε units of L.

- If a sequence $\{a_n\}$ agrees with a function f at every positive integer, and if $f(x)$ approaches a limit L as $x \to \infty$, the sequence must converge to the same limit L.

Theorem 9.1 (Limit of a sequence)

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x \to \infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n \to \infty} a_n = L.$$

Example 2 (Finding the limit of a sequence)

Find the limit of the sequence whose *n*th term is

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

- You learned that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.
- So, you can apply Theorem 9.1 to conclude that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e.$$

■

Theorem 9.2 (Properties of limits of sequences)

Let $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} b_n = K$.

1. *Scalar multiple :* $\lim_{n\to\infty} ca_n = cL$, *c is any real number*
2. *Sum or difference :* $\lim_{n\to\infty} (a_n \pm b_n) = L \pm K$
3. *Product :* $\lim_{n\to\infty}(a_n b_n) = LK$
4. *Quotient :* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{K}$, $b_n \neq 0$ *and* $K \neq 0$

Example 3 (Determining convergence or divergence)

a. Because the sequence $\{a_n\} = \{3 + (-1)^n\}$ has terms

$$2,\ 4,\ 2,\ 4,\ \ldots$$

that alternate between 2 and 4, the limit

$$\lim_{n\to\infty} a_n$$

does not exist. So, the sequence diverges.

b. For $\{b_n\} = \{\frac{n}{1-2n}\}$, divide the numerator and denominator by n to obtain

$$\lim_{n\to\infty} \frac{n}{1-2n} = \lim_{n\to\infty} \frac{1}{(1/n)-2} = -\frac{1}{2}$$

which implies that the sequence converges to $-\frac{1}{2}$. ■

Example 4 (Using L'Hôpital's Rule to determine convergence)

Show that the sequence whose nth term is $a_n = \frac{n^2}{2^n - 1}$ converges.

- Consider the function of a real variable

$$f(x) = \frac{x^2}{2^x - 1}.$$

- Applying L'Hôpital's Rule twice produces

$$\lim_{x\to\infty} \frac{x^2}{2^x - 1} = \lim_{x\to\infty} \frac{2x}{(\ln 2)2^x} = \lim_{x\to\infty} \frac{2}{(\ln 2)^2 2^x} = 0.$$

- Because $f(n) = a_n$ all for every positive integer, you can apply Theorem 9.1 to conclude that

$$\lim_{n\to\infty} \frac{n^2}{2^n - 1} = 0.$$

- So, the sequence converges to 0. ∎

- The symbol $n!$ (read "n factorial") is used to simplify some of the formulas. Let n be a positive integer; then n factorial is defined as $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$.
- As a special case, zero factorial is defined as $0! = 1$.
- From this definition, you can see that $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, and so on.
- Factorial follow the same conventions for order of operations as exponents. That is, $2n! = 2(n!)$ is different from $(2n)!$

Commonly used ordering If $a > 0$ and $b > 1$, then

$$\ln n \prec n^a \prec b^n \prec n!$$

where $a_n \prec b_n$ denotes that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

Theorem 9.3 (Squeeze Theorem for sequences)

If

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$$

and there exists an integer N such that $a_n \le c_n \le b_n$ for all $n > N$, then

$$\lim_{n\to\infty} c_n = L.$$

Example 5 (Using the Squeeze Theorem)

Show that the sequence $\{c_n\} = \left\{(-1)^n \frac{1}{n!}\right\}$ converges, and find its limit.

- To apply the Squeeze Theorem, you must find two convergent sequences that can be related to the given sequence.

- Two possibilities are $a_n = -1/2^n$ and $b_n = 1/2^n$, both of which converge to 0. By comparing the term $n!$ with 2^n, you can see that,

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots n = 24 \cdot \underbrace{5 \cdot 6 \cdots n}_{n-4 \text{ factors}} \qquad (n \geq 4)$$

and

$$2^n = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdots 2 = 16 \cdot \underbrace{2 \cdot 2 \cdots 2}_{n-4 \text{ factors}}. \qquad (n \geq 4)$$

- This implies that for $n \geq 4$, $2^n < n!$, and you have

$$\frac{-1}{2^n} \leq (-1)^n \frac{1}{n!} \leq \frac{1}{2^n}, \qquad n \geq 4$$

as shown in Figure 2.

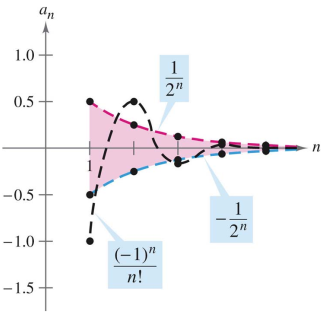


Figure 2: For $n \geq 4$, $(-1)^n/n!$ is squeezed between $-1/2^n$ and $1/2^n$.

- So, by the Squeeze Theorem it follows that

$$\lim_{n\to\infty}(-1)^n\frac{1}{n!}=0.$$

■

Theorem 9.4 (Absolute Value Theorem)

For the sequence $\{a_n\}$, *if*

$$\lim_{n\to\infty} |a_n| = 0 \qquad \textit{then} \qquad \lim_{n\to\infty} a_n = 0.$$

- Consider the two sequences $\{|a_n|\}$ and $\{-|a_n|\}$.
- Because both of these sequences converge to 0 and

$$-|a_n| \leq a_n \leq |a_n|$$

you can use the Squeeze Theorem to conclude that $\{a_n\}$ converges to 0. $\square$

Pattern recognition for sequences

- Sometimes the terms of a sequence are generated by some rule that does not explicitly identify the nth term of the sequence.
- In such cases, you may be required to discover a *pattern* in the sequence and to describe the nth term.
- Once the nth term has been specified, you can investigate the convergence or divergence of the sequence.

Example 6 (Finding the *n*th term of a sequence)

Find a sequence $\{a_n\}$ whose first five terms are

$$\frac{2}{1}, \frac{4}{3}, \frac{8}{5}, \frac{16}{7}, \frac{32}{9}, \ldots$$

and then determine whether the particular sequence you have chosen converges or diverges.

- First, note that the numerators are successive powers of 2, and the denominators form the sequence of positive odd integers.
- By comparing a_n with n, you have the following pattern.

$$\frac{2^1}{1}, \frac{2^2}{3}, \frac{2^3}{5}, \frac{2^4}{7}, \frac{2^5}{9}, \ldots, \frac{2^n}{2n-1}$$

- Using L'Hôpital's Rule to evaluate the limit of $f(x) = \frac{2^x}{(2x-1)}$, you obtain

$$\lim_{x\to\infty} \frac{2^x}{2x-1} = \lim_{x\to\infty} \frac{2^x(\ln 2)}{2} = \infty \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{2^n}{2n-1} = \infty.$$

- So, the sequence diverges.

The process of determining an nth term from the pattern observed in the first several terms of a sequence is an example of inductive reasoning.

Example 7 (Finding the nth term of a sequence)

Determine an nth term for a sequence whose first five terms are

$$-\frac{2}{1},\ \frac{8}{2},\ -\frac{26}{6},\ \frac{80}{24},\ -\frac{242}{120},\ \ldots$$

and then decide whether the sequence converges or diverges.

- Note that the numerators are 1 less than 3^n. So, you can reason that the numerators are given by the rule $3^n - 1$.
- Factoring the denominators produces

$$1 = 1 \quad 2 = 1\cdot 2 \quad 6 = 1\cdot 2\cdot 3 \quad 24 = 1\cdot 2\cdot 3\cdot 4 \quad 120 = 1\cdot 2\cdot 3\cdot 4\cdot 5$$

- This suggests that the denominators are represented by $n!$.

- Finally, because the signs alternate, you can write the nth term as

$$a_n = (-1)^n \left(\frac{3^n - 1}{n!} \right).$$

- From the discussion about the growth of $n!$, it follows that

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{3^n - 1}{n!} = 0.$$

- Applying Theorem 9.4, you can conclude that $\lim_{n\to\infty} a_n = 0$. So, the sequence $\{a_n\}$ converges to 0. ■

Monotonic sequences and bounded sequences

Definition 9.2 (Monotonic sequence)

A sequence $\{a_n\}$ is monotonic if its terms are nondecreasing

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$$

or if its terms are nonincreasing

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots .$$

Example 8 (Determining whether a sequence is monotonic)

Determine whether each sequence having the given nth term is monotonic.
a. $a_n = 3 + (-1)^n$ **b.** $b_n = \frac{2n}{1+n}$ **c.** $\frac{n^2}{2^n - 1}$

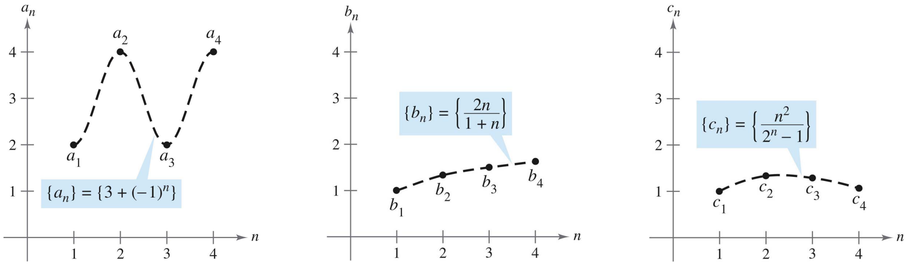
a. This sequence alternates between 2 and 4. So, it is not monotonic. See Figure 3

b. This sequence is monotonic because each successive term is larger than its predecessor. To see this, compare the terms b_n and b_{n+1}. [Note that, because n is positive, you can multiply each side of the inequality by $(1+n)$ and $(2+n)$ without reversing the inequality sign.]

$$\begin{aligned} b_{n+1} - b_n &= \frac{2(n+1)}{1+(n+1)} - \frac{2n}{1+n} = \frac{(1+n)(2n+2) - 2n(2+n)}{(n+2)(n+1)} \\ &= \frac{(2n^2+4n+2) - (2n^2+4n)}{(n+2)(n+1)} = \frac{2}{(n+2)(n+1)} > 0 \end{aligned}$$

Starting with the final inequality, which is valid, you can reverse the steps to conclude that the original inequality is also valid. See Figure 3

c. This sequence is not monotonic, because the second term is larger than the first term, and larger than the third. (Note that if you drop the first term, the remaining sequence $c_2, c_3, c_4, \ldots$ is monotonic. See Figure 3) ■



(a) Not monotonic. (b) Monotonic. (c) Not monotonic.

Figure 3: Graphically illustrates three sequences.

Definition 9.3 (Bounded sequence)

1. A sequence $\{a_n\}$ is bounded above if there is a real number M such that $a_n \leq M$ for all n. The number M is called an upper bound of the sequence.
2. A sequence a_n is bounded below if there is a real number N such that $N \leq a_n$ for all n. The number N is called a lower bound of the sequence.
3. A sequence $\{a_n\}$ is bounded if it is bounded above and bounded below.

- One important property of the real numbers is that they are complete. This means that there are no holes or gaps on the real number line. (The set of rational numbers does not have the completeness property.)

- The completeness axiom for real numbers can be used to conclude that if a sequence has an upper bound, it must have a least upper bound (an upper bound that is smaller than all other upper bounds for the sequence).
- For example, the least upper bound of the sequence $\{a_n\} = \{n/(n+1)\}$,

$$\frac{1}{2}, \ \frac{2}{3}, \ \frac{3}{4}, \ \frac{4}{5}, \ \ldots, \ \frac{n}{n+1}, \ \ldots$$

is 1.

Theorem 9.5 (Bounded monotonic sequences)

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

Example 9 (Bounded and monotonic sequences)

Determine whether or not the following sequences bounded or convergent.

a. $\{a_n\} = \left\{\frac{1}{n}\right\}$ **b.** $\{b_n\} = \left\{\frac{n^2}{(n+1)}\right\}$ **c.** $\{c_n\} = \{(-1)^n\}$

a. The sequence $\{a_n\} = \{1/n\}$ is both bounded and monotonic and so, by Theorem 9.5, must converge.

b. The divergent sequence $\{b_n\} = \left\{n^2/(n+1)\right\}$ is monotonic, but not bounded. (It is bounded below.)

c. The divergent sequence $\{c_n\} = \{(-1)^n\}$ is bounded, but not monotonic. ■

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Infinite series

- One important application of infinite sequences is in representing infinite summations.
- Informally, if $\{a_n\}$ is an infinite sequence, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots \qquad \text{Infinite series}$$

  is an infinite series (or simply a series).
- The numbers a_1, a_2, a_3, are the terms of the series.
- For some series it is convenient to begin the index at $n = 0$ (or some other integer).
- As a typesetting convention, it is common to represent an infinite series as simply $\sum a_n$.
- In such cases, the starting value for the index must be taken from the context of the statement.

- To find the sum of an infinite series, consider the following

$$S_1 = a_1 \qquad S_2 = a_1 + a_2 \qquad S_3 = a_1 + a_2 + a_3$$
$$S_4 = a_1 + a_2 + a_3 + a_4 \qquad S_5 = a_1 + a_2 + a_3 + a_4 + a_5 \qquad \cdots$$
$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

- If this sequence of partial sums converges, the series is said to converge.

Definition 9.4 (Convergent and divergent series)

For the infinite series $\sum_{n=1}^{\infty} a_n$ the nth partial sum is given by

$$S_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ converges. The limit S is called the sum of the series.

$$S = a_1 + a_2 + \cdots + a_n + \cdots \qquad S = \sum_{n=1}^{\infty} a_n$$

If $\{S_n\}$ diverges, then the series diverges.

Example 1 (Convergent and divergent series)

a. The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

has the following partial sums.

$$\begin{aligned}
S_1 &= \frac{1}{2} \\
S_2 &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\
S_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\
&\cdots \\
S_n &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}
\end{aligned}$$

- Because
$$\lim_{n\to\infty} \frac{2^n - 1}{2^n} = 1$$
it follow that the series converges and its sum is 1.

b. The nth partial sum of the series
$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$
is given by
$$S_n = 1 - \frac{1}{n+1}.$$

- Because the limit of S_n is 1, the series converges and its sum is 1.

c. The series
$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$$
diverges because $S_n = n$ and the sequence of partial sums diverges. ■

- The series $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$ is a telescoping series of the form

$$(b_1-b_2)+(b_2-b_3)+(b_3-b_4)+(b_4-b_5)+\cdots$$

  Note that b_2 is canceled by the second term, b_3 is canceled by the third term, and so on.

- Because the nth partial sum of this series is

$$S_n=b_1-b_{n+1}$$

  it follows that a telescoping series will converge if and only if b_n approaches a finite number as $n\to\infty$.

- Moreover, if the series converges, its sum is

$$S=b_1-\lim_{n\to\infty}b_{n+1}.$$

Example 2 (Writing a series in telescoping form)

Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{4n^2-1}$.

- Using partial fractions, you can write

$$a_n = \frac{2}{4n^2-1} = \frac{2}{(2n-1)(2n+1)} = \frac{1}{2n-1} - \frac{1}{2n+1}.$$

- From this telescoping form, you can see that the nth partial sum is

$$S_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = 1 - \frac{1}{2n+1}.$$

- So, the series converges and its sum is 1. That is,

$$\sum_{n=1}^{\infty} \frac{2}{4n^2-1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{2n+1}\right) = 1.$$

Geometric series

- The series $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ is a geometric series.
- In general, the series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, \qquad a \neq 0$$

is a geometric series with ratio r.

Theorem 9.6 (Convergence of a geometric series)

A geometric series with ratio r diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

Example 3 (Convergent and divergent geometric series)

a. The geometric series

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n = 3(1) + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots$$

has a ratio of $r = \frac{1}{2}$ with $a = 3$. Because $0 < |r| < 1$, the series converges and its sum is

$$S = \frac{a}{1-r} = \frac{3}{1-(1/2)} = 6.$$

b. The geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$$

has a ratio $r = \frac{3}{2}$. Because $|r| \geq 1$, the series diverges. ∎

Example 4 (A geometric series for a repeating decimal)

Use a geometric series to write $0.\overline{08}$ as the ratio of two integers.

- For the repeating decimal $0.\overline{08}$, you can write

$$0.080808\ldots = \frac{8}{10^2} + \frac{8}{10^4} + \frac{8}{10^6} + \frac{8}{10^8} + \cdots = \sum_{n=0}^{\infty} \left(\frac{8}{10^2}\right)\left(\frac{1}{10^2}\right)^n.$$

- For this series, you have $a = 8/10^2$ and $r = 1/10^2$. So,

$$0.080808\ldots = \frac{a}{1-r} = \frac{8/10^2}{1-(1/10^2)} = \frac{8}{99}.$$

- Try dividing 8 by 99 on a calculator to see that it produces $0.\overline{08}$. ■

Theorem 9.7 (Properties of infinite series)

Let $\sum a_n$ and $\sum b_n$ be convergent series, and let A, B, and c be real numbers. If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then the following series converge to the indicated sums.

1. $\sum_{n=1}^{\infty} ca_n = cA$
2. $\sum_{n=1}^{\infty}(a_n + b_n) = A + B$
3. $\sum_{n=1}^{\infty}(a_n - b_n) = A - B$

nth-term test for a convergent series

Theorem 9.8 (Limit of the nth term of a convergent series)

If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

- Assume that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = L.$$

- Then, because $S_n = S_{n-1} + a_n$ and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = L.$$

- It follows that

$$\begin{aligned} L &= \lim_{n\to\infty} S_n = \lim_{n\to\infty} (S_{n-1} + a_n) = \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty} a_n \\ &= L + \lim_{n\to\infty} a_n \end{aligned}$$

which implies that $\{a_n\}$ converges to 0. $\square$

- The contrapositive of Theorem 9.8 provides a useful test for divergence.
- This *n*th-Term Test for Divergence states that if the limit of the *n*th term of a series does not converge to 0, the series must diverge.

Theorem 9.9 (*n*th-term test for divergent)

If $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Example 5 (Using the *n*th-term test for divergent)

a. For the series $\sum_{n=0}^{\infty} 2^n$, you have

$$\lim_{n\to\infty} 2^n = \infty.$$

So the limit of the nth term is not 0, and the series diverges.

b. For the series $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$, you have

$$\lim_{n\to\infty} \frac{n!}{2n!+1} = \frac{1}{2}.$$

So, the limit of the nth term is not 0, and the series diverges.

c. For the series $\sum_{n=1}^{\infty} \frac{1}{n}$, you have

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Because the limit of the nth term is 0, the nth-term test for divergence does not apply and you can draw no conclusions about convergence or divergence. ■

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The Integral Test

Theorem 9.10 (The Integral Test)

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \textit{and} \quad \int_1^{\infty} f(x)\,\mathrm{d}x$$

either both converge or both diverge.

Example 1 (Using the Integral Test)

Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

- The function $f(x) = \frac{x}{(x^2+1)}$ is positive and continuous for $x \geq 1$.

- To determine whether f is decreasing, find the derivative.

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

- So, $f'(x) < 0$ for $x > 1$ and it follows that f satisfies the conditions for the Integral Test.
- You can integrate to obtain

$$\begin{aligned}
\int_1^\infty \frac{x}{x^2+1}\,\mathrm{d}x &= \frac{1}{2}\int_1^\infty \frac{2x}{x^2+1}\,\mathrm{d}x \\
&= \frac{1}{2}\lim_{b\to\infty}\int_1^b \frac{2x}{x^2+1}\,\mathrm{d}x = \frac{1}{2}\lim_{b\to\infty}\left[\ln(x^2+1)\right]_1^b \\
&= \frac{1}{2}\lim_{b\to\infty}\left[\ln(b^2+1) - \ln 2\right] = \infty.
\end{aligned}$$

■

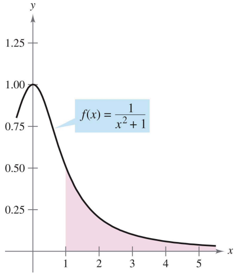
Example 2 (Using the Integral Test)

Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

- Because $f(x) = 1/(x^2+1)$ satisfies the conditions for the Integral Test (check this), you can integrate to obtain

$$\int_1^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = \lim_{b\to\infty}\int_1^{b} \frac{1}{x^2+1}\,\mathrm{d}x = \lim_{b\to\infty}[\arctan x]_1^b$$
$$= \lim_{b\to\infty}(\arctan b - \arctan 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

- So, the series converges. ■



- A second type of series has a simple arithmetic test for convergence or divergence.
- A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

  is a p-series, where p is a positive constant.
- For $p = 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

  is the harmonic series.
- A general harmonic series is of the form $\sum \frac{1}{(an+b)}$.
- In music, strings of the same material, diameter, and tension, whose lengths form a harmonic series, produce harmonic tones.

Euler-Mascheroni constant γ (C)

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) \approx 0.5772156649$$

is a mathematical constant recurring in analysis and number theory.

Riemann zeta function $\zeta(s)$

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

is a function of a complex variable s that analytically continues the sum of the infinite series which converges when the real part of s is greater than 1. The Riemann zeta function plays a pivotal role in analytic number theory and has applications in physics, probability theory, and applied statistics.

Theorem 9.11 (Convergence of p series)

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

1. *converges if $p > 1$, and* **2.** *diverges if $0 < p \leq 1$.*

- The proof follows from the Integral Test and from Theorem 8.7, which states that

$$\int_1^{\infty} \frac{1}{x^p} \, \mathrm{d}x$$

converges if $p > 1$ and diverges if $0 < p \leq 1$. □

Example 3 (Convergent and divergent p series)

Discuss the convergence or divergence of

a. the harmonic series and **b.** the p-series with $p = 2$.

a. From Theorem 9.11, it follows that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges.

b. From Theorem 9.11, it follows that the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

converges. ∎

Example 4 (Testing a series for convergence)

Determine whether the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

- This series is similar to the divergent harmonic series.
- If its terms were larger than those of the harmonic series, you would expect it to diverge.
- The function $f(x) = 1/(x \ln x)$ is positive and continuous for $x \geq 2$.
- To determine whether f is decreasing, first rewrite f as $f(x) = (x \ln x)^{-1}$ and then find its derivative.

$$f'(x) = (-1)(x \ln x)^{-2}(1 + \ln x) = -\frac{1 + \ln x}{x^2(\ln x)^2}$$

- So, $f'(x) < 0$ for $x > 2$ and it follows that f satisfies the conditions for the Integral Test.

$$\int_2^\infty \frac{1}{x \ln x}\,\mathrm{d}x = \int_2^\infty \frac{1/x}{\ln x}\,\mathrm{d}x = \lim_{b\to\infty} [\ln(\ln x)]_2^b$$
$$= \lim_{b\to\infty} [\ln(\ln b) - \ln(\ln 2)] = \infty$$

- The series diverges. ■

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Direct comparison test

- For the convergence tests the terms of the series have to be fairly simple and the series must have special characteristics in order for the convergence tests to be applied.
- A slight deviation from these special characteristics can make a test nonapplicable.
- For example, in the following pairs, the second series cannot be tested by the same convergence test as the first series even though it is similar to the first.
  1. $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is geometric, but $\sum_{n=0}^{\infty} \frac{n}{2^n}$ is not.
  2. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series, but $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ is not.
  3. $a_n = \frac{n}{(n^2+3)^2}$ is easily integrated, but $b_n = \frac{n^2}{(n^2+3)^2}$.

Theorem 9.12 (Direct Comparison Test)

Let $0 < a_n \leq b_n$ for all n.

1. *If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

2. *If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.*

- To prove the first property, let $L = \sum_{n=1}^{\infty} b_n$ and let

$$S_n = a_1 + a_2 + \cdots + a_n.$$

- Because $0 < a_n \leq b_n$, the sequence $S_1, S_2, S_3, \ldots$ is nondecreasing and bounded above by L; so, it must converge. Because

$$\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n$$

it follows that $\sum a_n$ converges. The second property is logically equivalent to the first. $\square$

Example 1 (Using the Direct Comparison Test)

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$.

- This series resembles

$$\sum_{n=1}^{\infty} \frac{1}{3^n}. \qquad \text{Convergent geometric series}$$

- Term-by-term comparison yields

$$a_n = \frac{1}{2+3^n} < \frac{1}{3^n} = b_n, \quad n \geq 1.$$

- So, by the Direct Comparison Test, the series converges. ■

Example 2 (Using the Direct Comparison Test)

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$.

- This series resembles

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}. \qquad \text{Divergent } p\text{-series}$$

- Term-by-term comparison yields

$$\frac{1}{2+\sqrt{n}} \leq \frac{1}{\sqrt{n}}, \quad n \geq 1$$

which does not meet the requirements for divergence. (Remember that if term-by-term comparison reveals a series that is smaller than a divergent series, the Direct Comparison Test tells you nothing.)

- Still expecting the series to diverge, you can compare the given series with

$$\sum_{n=1}^{\infty} \frac{1}{n}. \qquad \text{Divergent harmonic series}$$

- In this case, term-by-term comparison yields

$$a_n = \frac{1}{n} \leq \frac{1}{2+\sqrt{n}} = b_n, \quad n \geq 4$$

and, by the Direct Comparison Test, the given series diverges. ∎

Remember that both parts of the Direct Comparison Test require that $0 < a_n \leq b_n$. Informally, the test says the following about the two series with nonnegative terms.

1. If the "larger" series converges, the "smaller" series must also converge.
2. If the "smaller" series diverges, the "larger" series must also diverge.

Limit comparison test

- Often a given series closely resembles a p-series or a geometric series, yet you cannot establish the term-by-term comparison necessary to apply the Direct Comparison Test. Under these circumstances you may be able to apply a second comparison test, called the Limit Comparison Test.

Theorem 9.13 (Limit Comparison Test)

Suppose that $a_n > 0$, $b_n > 0$, and

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$

where L is finite and positive. Then the two series $\sum a_n$ and $\sum b_n$ either both converge both diverge.

- Because $a_n > 0$, $b_n > 0$, and

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$

there exists $N > 0$ such that

$$0 < \frac{a_n}{b_n} < L + 1, \quad \text{for } n \geq N.$$

- This implies that

$$0 < a_n < (L+1)b_n.$$

- So, by the Direct Comparison Test, the convergence of $\sum b_n$ implies the convergence $\sum a_n$ and the divergence of $\sum a_n$ implies the divergence $\sum b_n$.
- Similarly, the fact that

$$\lim_{n\to\infty}\left(\frac{b_n}{a_n}\right) = \frac{1}{L}$$

can be used to show that the convergence of $\sum a_n$ all implies the convergence of $\sum b_n$ and the divergence of $\sum b_n$ implies the divergence $\sum a_n$.

$\square$

Example 3 (Using the Limit Comparison Test)

Show that the following general harmonic series diverges.

$$\sum_{n=1}^{\infty} \frac{1}{an+b}, \quad a > 0,\ b > 0$$

- By comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$ (divergent harmonic series), you have

$$\lim_{n\to\infty} \frac{1/(an+b)}{1/n} = \lim_{n\to\infty} \frac{n}{an+b} = \frac{1}{a}.$$

- Because this limit is greater than 0, you can conclude from the Limit Comparison Test that the given series diverges. ■

- The Limit Comparison Test works well for comparing a "messy" algebraic series with a p-series.

Given Series	Comparison Series	Conclusion
$\sum_{n=1}^{\infty} \frac{1}{3n^2-4n+5}$	$\sum_{n=1}^{\infty} \frac{1}{n^2}$	Both series converge.
$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-2}}$	$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$	Both series diverge.
$\sum_{n=1}^{\infty} \frac{n^2-10}{4n^5+n^3}$	$\sum_{n=1}^{\infty} \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3}$	Both series converge.

- When choosing a series for comparison, you can disregard all but the highest powers of n in both the numerator and the denominator.

Example 4 (Using the Limit Comparison Test)

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$.

- Disregarding all but the highest powers of n in the numerator and the denominator, you can compare the series with

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}. \qquad \text{Convergent } p\text{-series}$$

- Because

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\sqrt{n}}{n^2+1}\right)\left(\frac{n^{3/2}}{1}\right) = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

you can conclude by the Limit Comparison Test that the given series converges. ∎

Example 5 (Using the Limit Comparison Test)

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$.

- A reasonable comparison would be with the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}. \qquad \text{Divergent series}$$

- Note that this series diverges by the nth-Term Test. From the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n2^n}{4n^3+1}\right)\left(\frac{n^2}{2^n}\right) = \lim_{n\to\infty} \frac{1}{4+(1/n^3)} = \frac{1}{4}$$

you can conclude that the given series diverges. ■

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Alternating series

- The simplest series that contain both positive and negative terms is an alternating series, whose terms alternate in sign. For example, the geometric series

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n}=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots$$

  is an alternating geometric series with $r=-1/2$.
- Alternating series occur in two ways: either the odd terms are negative or the even terms are negative.

Theorem 9.14 (Alternating Series Test)

Let $a_n > 0$. *The alternating series*

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

converge if the following two conditions are met.

1. $\lim_{n\to\infty} a_n = 0$ **2.** $a_{n+1} \leq a_n$, *for all* n

Remark

The second condition in the Alternating Series Test can be modified to require only that $0 < a_{n+1} \leq a_n$ for all n greater than some **integer** N.

Example 1 (Using the Alternating Comparison Test)

Determine the convergence or divergence of $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$.

- Note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$. So, the first condition of Theorem 9.14 is satisfied.
- Also note that the second condition of Theorem 9.14 is satisfied because

$$a_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = a_n$$

for all n. So, applying the Alternating Series Test, you can conclude that the series converges. ■

Example 2 (Using the Alternating Series Test)

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}}$.

- To apply the Alternating Series Test, note that, for $n \geq 1$,

$$\frac{1}{2} \leq \frac{n}{n+1} \quad \frac{2^{n-1}}{2^n} \leq \frac{n}{n+1} \quad (n+1)2^{n-1} \leq n2^n \quad \frac{n+1}{2^n} \leq \frac{n}{2^{n-1}}.$$

- So, $a_{n+1} = (n+1)2^n \leq n/2^{n-1} = a_n$ for all n.
- Furthermore, by L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{x}{2^{x-1}} = \lim_{x\to\infty} \frac{1}{2^{x-1}(\ln 2)} = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{n}{2^{n-1}} = 0.$$

- Therefore, by the Alternating Series Test, the series converges. ■

Example 3 (When the Alternating Series Test does not apply)

a. The alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n} = \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \cdots$$

passes the second condition of the Alternating Series Test because $a_{n+1} \leq a_n$ for all n.

- You cannot apply the Alternating Series Test, however, because the series does not pass the first condition ($\lim_{n\to\infty} a_n = 1 \neq 0$). In fact, the series diverges.

b. The alternating series

$$\frac{2}{1} - \frac{1}{1} + \frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \cdots$$

passes the first condition because all approaches 0 as $n \to \infty$.

- You cannot apply the Alternating Series Test, however, because the series does not pass the second condition.
- To conclude that the series diverges, you can argue that S_{2N} equals the Nth partial sum of the divergent harmonic series.
- This implies that the sequence of partial sums diverges. So, the series diverges. $\blacksquare$

- For a convergent alternating series, the partial sum S_N can be a useful approximation for the sum S of the series. The error involved in using $S \approx S_N$ is the remainder $R_N = S - S_N$.

Theorem 9.15 (Alternating Series Remainder)

If a convergent alternating series satisfies the condition $a_{n+1} \leq a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \leq a_{N+1}.$$

Example 4 (Approximating the sum of an alternating series)

Approximate the sum of the following series by its first six terms.

$$1 - e^{-1} = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n!} \right) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \cdots$$

- The series converges by the Alternating Series Test because

$$\frac{1}{(n+1)!} \leq \frac{1}{n!} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n!} = 0.$$

- The sum of the first six terms is

$$S_6 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} = \frac{91}{144} \approx 0.63194$$

  and, by the Alternating Series Remainder, you have

$$|S - S_6| = |R_6| \leq a_7 = \frac{1}{5040} \approx 0.0002.$$

- So, the sum S lies between $0.63194 - 0.0002$ and $0.63194 + 0.0002$,

$$0.63174 \leq S \leq 0.63214.$$

Absolute and conditional convergence

- Occasionally, a series may have both positive and negative terms and not be an alternating series. For instance, the series

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$$

  has both positive and negative terms, yet it is not an alternating series.
- One way to obtain some information about the convergence of this series is to investigate the convergence of the series

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|.$$

- By direct comparison, you have $|\sin n| \leq 1$ for all n, so

$$\left| \frac{\sin n}{n^2} \right| \leq \frac{1}{n^2}, \quad n \geq 1.$$

- Therefore, by the Direct Comparison Test, the series $\sum \left|\frac{\sin n}{n^2}\right|$ converges.

Theorem 9.16 (Absolute convergence)

If the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges.

- The converse of Theorem 9.16 is not true. For instance, the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges by the Alternating Series Test. Yet the harmonic series diverges. This type of convergence is called conditional.

Definition 9.5 (Absolute and conditional convergence)

1. $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.
2. $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example 6 (Absolute and conditional convergence)

Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n} = \frac{0!}{2^0} - \frac{1!}{2^1} + \frac{2!}{2^2} - \frac{3!}{2^3} + \cdots$

b. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = -\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \cdots$

a. By the *n*th-term test for divergence, you can conclude that this series diverges.

b. The given series can be shown to be convergent by the Alternating Series Test. Moreover, because the p-series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

diverges, the given series is conditionally convergent. ■

Example 7 (Absolute and conditional convergence)

Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{3^n} = -\frac{1}{3} - \frac{1}{9} + \frac{1}{27} + \frac{1}{81} - \cdots$

b. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)} = -\frac{1}{\ln 2} + \frac{1}{\ln 3} - \frac{1}{\ln 4} + \frac{1}{\ln 5} - \cdots$

a. This is not an alternating series. However, because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n(n+1)/2}}{3^n} \right| = \sum_{n=1}^{\infty} \frac{1}{3^n}$$

is a convergent geometric series, you can apply Theorem 9.16 to conclude that the given series is absolutely convergent (and therefore convergent).

b. In this case, the Alternating Series Test indicates that the given series converges. However, the series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\ln(n+1)} \right| = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \cdots$$

diverges by direct comparison with the terms of the harmonic series. Therefore, the given series is conditionally convergent. ■

- A finite sum such as $(1 + 3 - 2 + 5 - 4)$ can be rearranged without changing the value of the sum. This is not necessarily true of an infinite series—it depends on whether the series is absolutely convergent (every rearrangement has the same sum) or conditionally convergent.

Example 8 (Rearrangement of a series)

The alternating harmonic series converges to $\ln 2$. That is,

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

Rearrange the series to produce a different sum.

- Consider the following rearrangement.

$$
\begin{aligned}
&1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\frac{1}{7}-\frac{1}{14}-\cdots\\
&=\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\left(\frac{1}{7}-\frac{1}{14}\right)-\cdot\\
&=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\cdots\\
&=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots\right)=\frac{1}{2}(\ln 2)
\end{aligned}
$$

- By rearranging the terms, you obtain a sum that is half the original sum. ■

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The Ratio Test

- This section begins with a test for absolute convergence—the Ratio Test.

Theorem 9.17 (Ratio Test)

Let $\sum a_n$ be a series with nonzero terms.

1. $\sum a_n$ *converges absolutely if* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.
2. $\sum a_n$ *diverges if* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ *or* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$.
3. *The Ratio Test is inconclusive if* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

Remark

Although the Ratio Test is not a cure for all ills related to testing for convergence, it is particularly useful for series that converge rapidly. Series involving factorials or exponentials are frequently of this type.

Example 1 (Using the Ratio Test)

Determine the convergence or divergence of $\sum_{n=0}^{\infty} \frac{2^n}{n!}$.

- Because $a_n = 2^n/n!$, you can write the following.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!} \right]$$
$$= \lim_{n\to\infty} \left[\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right] = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$$

- This series converges because the limit of $|a_{n+1}/a_n|$ is less than 1. ■

Example 2 (Using the Ratio Test)

Determine whether each series converges or diverges.

a. $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ **b.** $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

a. This series converges because the limit of $|a_{n+1}/a_n|$ is less than 1.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[(n+1)^2 \left(\frac{2^{n+2}}{3^{n+1}} \right) \left(\frac{3^n}{n^2 2^{n+1}} \right) \right]$$
$$= \lim_{n\to\infty} \frac{2(n+1)^2}{3n^2} = \frac{2}{3} < 1$$

b. This series diverges because the limit of $|a_{n+1}/a_n|$ is greater than 1.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\frac{(n+1)^{n+1}}{(n+1)!} \left(\frac{n!}{n^n} \right) \right] = \lim_{n\to\infty} \left[\frac{(n+1)^{n+1}}{n+1} \left(\frac{1}{n^n} \right) \right]$$
$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n = e > 1 \qquad \blacksquare$$

Example 3 (A failure of the Ratio Test)

Determine the convergence or divergence of $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+1}$.

- The limit of $|a_{n+1}/a_n|$ is equal to 1.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\left(\frac{\sqrt{n+1}}{n+2}\right)\left(\frac{n+1}{\sqrt{n}}\right)\right]$$
$$= \lim_{n\to\infty}\left[\sqrt{\frac{n+1}{n}}\left(\frac{n+1}{n+2}\right)\right] = \sqrt{1}(1) = 1$$

- So, the Ratio Test is inconclusive.
- To determine whether the series converges, you need to try a different test.
- In this case, you can apply the Alternating Series Test. To show that $a_{n+1} \le a_n$, let

$$f(x) = \frac{\sqrt{x}}{x+1}.$$

- Then the derivative is

$$f'(x) = \frac{-x+1}{2\sqrt{x}(x+1)^2}.$$

- Because the derivative is negative for $x > 1$, you know that f is a decreasing function.
- Also, by L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{\sqrt{x}}{x+1} = \lim_{x\to\infty} \frac{1/\left(2\sqrt{x}\right)}{1} = \lim_{x\to\infty} \frac{1}{2\sqrt{x}} = 0.$$

- Therefore, by the Alternating Series Test, the series converges. ■

The Root Test

- The next test for convergence or divergence of series works especially well for series involving nth powers.

Theorem 9.18 (Root Test)

Let $\sum a_n$ be a series.

1. *$\sum a_n$ converges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.*
2. *$\sum a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$.*
3. *The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.*

Remark

The Root Test is always inconclusive for any p-series.

Example 4 (Using the Root Test)

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$.

- You can apply the Root Test as follows.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{e^{2n}}{n^n}} = \lim_{n\to\infty} \frac{e^{2n/n}}{n^{n/n}} = \lim_{n\to\infty} \frac{e^2}{n} = 0 < 1$$

- Because this limit is less than 1, you can conclude that the series converges absolutely (and therefore converges). ∎

Strategies for testing series

- Does the nth term approach 0? If not, the series diverges.
- Is the series one of the special types—geometric, p-series, telescoping, or alternating?
- Can the Integral Test, the Root Test, or the Ratio Test be applied?
- Can the series be compared favorably to one of the special types?

Example 5 (Applying the strategies for testing series)

Determine the convergence or divergence of

a. $\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$ **b.** $\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$ **c.** $\sum_{n=1}^{\infty} ne^{-n^2}$

d. $\sum_{n=1}^{\infty} \frac{1}{3n+1}$ **e.** $\sum_{n=1}^{\infty}(-1)^n \frac{3}{4n+1}$ **f.** $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

g. $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$

a. For this series, the limit of the nth term is not 0 ($a_n \to 1/3$ as $n \to \infty$). So, by the nth-Term Test, the series diverges.

b. This series is geometric. Moreover, because the ratio $r = \pi/6$ of the terms is less than 1 in absolute value, you can conclude that the series converges.

c. Because the function $f(x) = xe^{-x^2}$ is easily integrated, you can use the Integral Test to conclude that the series converges.

d. The nth term of this series can be compared to the nth term of the harmonic series. After using the Limit Comparison Test, you can conclude that the series diverges.

e. This is an alternating series whose nth term approaches 0. Because $a_{n+1} \leq a_n$. you can use the Alternating Series Test to conclude that the series converges.

f. The nth term of this series involves a factorial, which indicates that the Ratio Test may work well. After applying the Ratio Test, you can conclude that the series diverges.

g. The nth term of this series involves a variable that is raised to the nth power, which indicates that the Root Test may work well. After applying the Root Test, you can conclude that the series converges. ■

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Polynomial approximations of elementary functions

- To find a polynomial function P that approximates another function f, begin by choosing a number c in the domain of f at which f and P have the same value. That is,

$$P(c) = f(c). \qquad \text{Graphs of } f \text{ and } P \text{ pass through } (c, f(c))$$

- The approximating polynomial is said to be expanded about c or centered at c.
- Geometrically, the requirement that $P(c) = f(c)$ means that the graph of P passes through the point $(c, f(c))$. Of course, there are many polynomials whose graphs pass through the point $(c, f(c))$.

- To find a polynomial whose graph resembles the graph of f near this point. One way to do this is to impose the additional requirement that the slope of the polynomial function be the same as the slope of the graph of f at the point $(c, f(c))$.

$$P'(c) = f'(c). \qquad \text{Graphs of } f \text{ and } P \text{ have the same at } (c, f(c))$$

- With these two requirements, you can obtain a simple linear approximation of f, as shown in Figure 5.

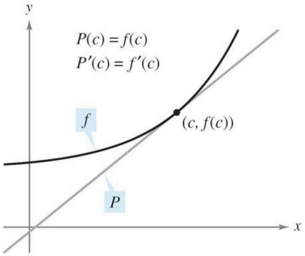


Figure 5: Near $(c, f(c))$, the graph of P can be used to approximate the graph of f.

Example 1 (First-degree polynomial approximation of $f(x) = e^x$)

For the function $f(x) = e^x$, find a first-degree polynomial function

$$P_1(x) = a_0 + a_1 x$$

whose value and slope agree with the value and slope of f at $x = 0$.

- Because $f(x) = e^x$ and $f'(x) = e^x$, the value and the slope of f, at $x = 0$, are given by

$$f(0) = e^0 = 1 \quad \text{and} \quad f'(0) = e^0 = 1.$$

- Because $P_1(x) = a_0 + a_1 x$, you can use the condition that $P_1(0) = f(0)$ to conclude that $a_0 = 1$.
- Moreover, because $P_1'(x) = a_1$, you can use the condition that $P_1'(0) = f'(0)$ to conclude that $a_1 = 1$.
- Therefore, $P_1(x) = 1 + x$. Figure 6 shows the graphs of $P_1(x) = 1 + x$ and $f(x) = e^x$. ∎

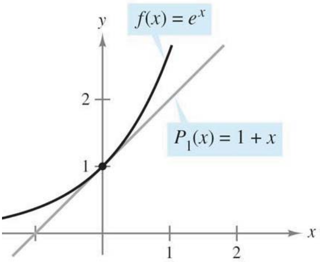


Figure 6: $P_1(x) = 1 + x$ is the first-degree polynomial approximation of $f(x) = e^x$.

Example 2 (Third-degree polynomial approximation of $f(x) = e^x$)

Construct a table comparing the values of the polynomial

$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3$$

with $f(x) = e^x$ for several values of x near 0.

- Using a calculator or a computer, you can obtain the results shown in the table.
- Note that for $x = 0$, the two functions have the same value, but that as x moves farther away from 0, the accuracy of the approximating polynomial $P_3(x)$ decreases. ■

x	-1.0	-0.2	-0.1	0	0.1	0.2	1.0
e^x	0.3679	0.81873	0.904837	1	1.105171	1.22140	2.7183
$P_3(x)$	0.3333	0.81867	0.904833	1	1.105167	1.22133	2.6667

Taylor and Maclaurin polynomials

- The polynomial approximation of $f(x) = e^x$ is expanded about $c = 0$. For expansions about an arbitrary value of c, it is convenient to write the polynomial in the form

$$P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n.$$

- In this form, repeated differentiation produces

$$\begin{aligned}
P_n'(x) &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots + na_n(x-c)^{n-1} \\
P_n''(x) &= 2a_2 + 2(3a_3)(x-c) + \cdots + n(n-1)a_n(x-c)^{n-2} \\
P_n'''(x) &= 2(3a_3) + \cdots + n(n-1)(n-2)a_n(x-c)^{n-3} \\
&\ \ \vdots \\
P_n^{(n)}(x) &= n(n-1)(n-2)\cdots(2)(1)a_n.
\end{aligned}$$

- Letting $x = c$, you then obtain

$$P_n(c) = a_0, \quad P_n'(c) = a_1, \quad P_n''(c) = 2a_2, \quad \ldots, \quad P_n^{(n)}(c) = n!a_n.$$

and because the values of f and its first n derivatives must agree with the values of P_n and its first n derivatives at $x = c$, it follows that

$$f(c) = a_0, \quad f'(c) = a_1, \quad \frac{f''(c)}{2!} = a_2, \quad \ldots, \quad \frac{f^{(n)}(c)}{n!} = a_n.$$

- With these coefficients, you can obtain the following definition of Taylor polynomials, named after the English mathematician Taylor, Brook (1685-1731), and
Maclaurin polynomials, named after the English mathematician Maclaurin, Colin (1698-1746).

Definition 9.6 (Taylor polynomial and Maclaurin polynomial)

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the nth Taylor polynomial for f at c. If $c = 0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

is also called the nth Maclaurin polynomial for f at c.

Example 3 (A Maclaurin polynomial for $f(x) = e^x$)

Find the nth Maclaurin polynomial for $f(x) = e^x$.

- The nth Maclaurin polynomial for

$$f(x) = e^x$$

is given by

$$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n.$$

■

Example 4 (Finding Taylor polynomials for $\ln x$)

Find the Taylor polynomials P_0, P_1, P_2, P_3, and P_4, for $\ln x$ centered at $c = 1$.

- Expanding about $c = 1$ yields the following.

$$\begin{aligned} f(x) &= \ln x & f(1) &= \ln 1 = 0 \\ f'(x) &= \frac{1}{x} & f'(1) &= \frac{1}{1} = 1 \\ f''(x) &= -\frac{1}{x^2} & f''(1) &= -\frac{1}{1^2} = -1 \\ f'''(x) &= \frac{2!}{x^3} & f'''(1) &= \frac{2!}{1^3} = 2 \\ f^{(4)}(x) &= -\frac{3!}{x^4} & f^{(4)}(1) &= -\frac{3!}{1^4} = -6 \end{aligned}$$

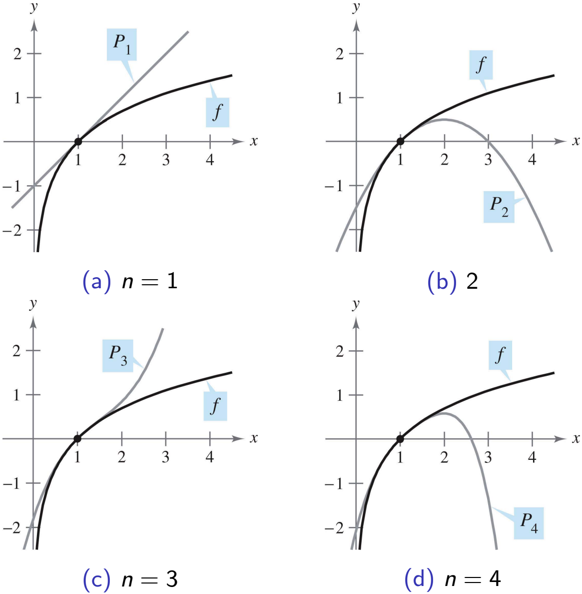
- Therefore, the Taylor polynomials are as follows.

$$P_0(x) = f(1) = 0$$
$$P_1(x) = f(1) + f'(1)(x-1) = (x-1)$$
$$P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2$$

- and

$$\begin{aligned} P_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \\ P_4(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &\quad + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \end{aligned}$$

- Figure 7 compares the graphs of P_1, P_2, P_3, and P_4 with the graph of $f(x) = \ln x$. Note that near $x = 1$ the graphs are nearly indistinguishable. For instance, $P_4(1.1) \approx 0.0953083$ and $\ln(1.1) \approx 0.0953102$. ■



(a) $n = 1$ (b) 2
(c) $n = 3$ (d) $n = 4$

Figure 7: As n increases, the graph of P_n, becomes a better and better approximation of the graph of $f(x) = \ln x$ near $x = 1$.

Example 5 (Finding Maclaurin polynomials for $\cos x$)

Find the Maclaurin polynomials P_0, P_2, P_4, and P_6 for $f(x) = \cos x$. Use $P_6(x)$ to approximate the value of $\cos(0.1)$.

- Expanding about $c = 0$ yields the following.

$$\begin{aligned} f(x) &= \cos x & \qquad f(0) &= \cos 0 = 1 \\ f'(x) &= -\sin x & f'(0) &= -\sin 0 = 0 \\ f''(x) &= -\cos x & f''(0) &= -\cos 0 = -1 \\ f'''(x) &= \sin x & f'''(0) &= \sin 0 = 0 \end{aligned}$$

- Through repeated differentiation, you can see that the pattern 1, 0, -1, 0 continues, and you obtain the following Maclaurin polynomials.

$$\begin{aligned} P_0(x) &= 1, & P_2(x) &= 1 - \frac{1}{2!}x^2, \\ P_4(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4, & P_6(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \end{aligned}$$

- Using $P_6(x)$, you obtain the approximation $\cos(0.1) \approx 0.995004165$, which coincides with the calculator value to nine decimal places. Figure 8 compares the graphs of $f(x) = \cos x$ and P_6. ■

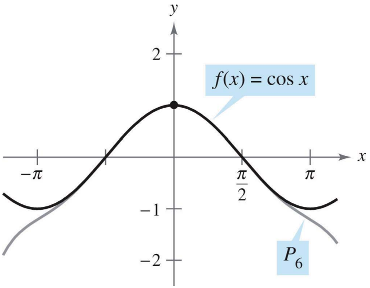


Figure 8: Near $(0, 1)$, the graph of P_6 can be used to approximate the graph of $f(x) = \cos x$.

Example 6 (Finding a Taylor polynomial for $\sin x$)

Find the third Taylor polynomial for $f(x) = \sin x$, expanded about $c = \pi/6$.

- Expanding about $c = \pi/6$ yields the following.

$$
\begin{aligned}
f(x) &= \sin x & f\left(\frac{\pi}{6}\right) &= \sin\frac{\pi}{6} = \frac{1}{2} \\
f'(x) &= \cos x & f'\left(\frac{\pi}{6}\right) &= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \\
f''(x) &= -\sin x & f''\left(\frac{\pi}{6}\right) &= -\sin\frac{\pi}{6} = -\frac{1}{2} \\
f'''(x) &= -\cos x & f'''\left(\frac{\pi}{6}\right) &= -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}
\end{aligned}
$$

- So, the third Taylor polynomial for $f(x) = \sin x$, expanded about $c = \pi/6$, is

$$
\begin{aligned}
P_3(x) &= f(\pi/6) + f'(\pi/6)(x - \pi/6) + \frac{f''(\pi/6)}{2!}(x - \pi/6)^2 \\
&\quad + \frac{f'''(\pi/6)}{3!}(x - \pi/6)^3 \\
&= \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \pi/6) - \frac{1}{2(2!)}(x - \pi/6)^2 - \frac{\sqrt{3}}{2(3!)}(x - \pi/6)^3.
\end{aligned}
$$

- Figure 9 compares the graphs of $f(x) = \sin x$ and P_3. ■

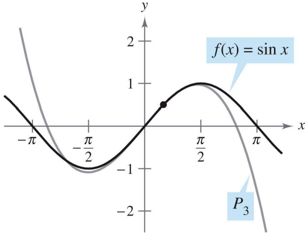


Figure 9: Near $(\pi/6, 1/2)$, the graph of P_3 can be used to approximate the graph of $f(x) = \sin x$.

Example 7 (Approximation using Maclaurin polynomials)

Use a fourth Maclaurin polynomial to approximate the value of ln(1.1).

- Because 1.1 is closer to 1 than to 0, you should consider Maclaurin polynomials for the function $g(x) = \ln(1+x)$.

$$\begin{aligned}
g(x) &= \ln(1+x) & g(0) &= \ln(1+0) = 0 \\
g'(x) &= (1+x)^{-1} & g'(0) &= (1+0)^{-1} = 1 \\
g''(x) &= -(1+x)^{-2} & g''(0) &= -(1+0)^{-2} = -1 \\
g'''(x) &= 2(1+x)^{-3} & g'''(0) &= 2(1+0)^{-3} = 2 \\
g^{(4)}(x) &= -6(1+x)^{-4} & g^{(4)}(0) &= -6(1+0)^{-4} = -6
\end{aligned}$$

- Note that you obtain the same coefficients as in Example 4. Therefore, the fourth Maclaurin polynomial for $g(x) = \ln(1+x)$ is

$$\begin{aligned} P_4(x) &= g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + \frac{g^{(4)}(0)}{4!}x^4 \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4. \end{aligned}$$

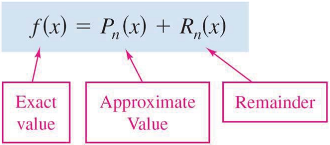
- Consequently,

$$\ln(1.1) = \ln(1+0.1) \approx P_4(0.1) \approx 0.0953083.$$

Check to see that the fourth Taylor polynomial (from Example 4), evaluated at $x = 1.1$, yields the same result. ■

Remainder of a Taylor polynomial

- An approximation technique is of little value without some idea of its accuracy.
- To measure the accuracy of approximating a function value $f(x)$ by the Taylor polynomial $P_n(x)$, you can use the concept of a remainder $R_n(x)$, defined as follows.



- So, $R_n(x) = f(x) - P_n(x)$. The absolute value of $R_n(x)$ is called the error associated with the approximation. That is,

$$\text{Error} = |R_n(x)| = |f(x) - P_n(x)|$$

- The next theorem gives a general procedure for estimating the remainder associated with a Taylor polynomial.

- This important theorem is called Taylor's Theorem, and the remainder given in the theorem is called the Lagrange form of the remainder.

Theorem 9.19 (Taylor's Theorem)

If a function f is differentiable through order $n+1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

Example 8 (Determining the accuracy of an approximation)

The third Maclaurin polynomial for $\sin x$ is given by

$$P_3(x) = x - \frac{x^3}{3!}.$$

Use Taylor's Theorem to approximate $\sin(0.1)$ by $P_3(0.1)$ and determine the accuracy of the approximation.

- Using Taylor's Theorem, you have

$$\sin x = x - \frac{x^3}{3!} + R_3(x) = x - \frac{x^3}{3!} + \frac{f^{(4)}(z)}{4!}x^4$$

  where $0 < z < 0.1$.

- Therefore,

$$\sin(0.1) \approx 0.1 - \frac{(0.1)^3}{3!} \approx 0.1 - 0.000167 = 0.099833.$$

- Because $f^{(4)}(z) = \sin z$, it follows that the error $|R_3(0.1)|$ can be bounded as follows.

$$0 < R_3(0.1) = \frac{\sin z}{4!}(0.1)^4 < \frac{0.0001}{4!} \approx 0.000004.$$

- This implies that

$$0.099833 < \sin(0.1) = 0.099833 + R_3(x) < 0.099833 + 0.000004$$
$$0.099833 < \sin(0.1) < 0.099837. \quad \blacksquare$$

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- An important function $f(x) = e^x$ can be represented exactly by an infinite series called a power series. For example, the power series representation for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots .$$

- For each real number x, it can be shown that the infinite series on the right converges to the number e^x.

Definition 9.7 (Power series)

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a power series. More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots$$

is called a power series centered at c, where c is a constant.

Example 1 (Power series)

a. The following power series is centered at 0.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

b. The following power series is centered at -1.

$$\sum_{n=0}^{\infty} (-1)^n (x+1)^n = 1 - (x+1) + (x+1)^2 - (x+1)^3 + \cdots$$

c. The following power series is centered at 1.

$$\sum_{n=1}^{\infty} \frac{1}{n}(x-1)^n = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots$$

■

Radius and interval of convergence

A power series in x can be viewed as a function of x

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

where the domain of f is the set of all x for which the it converges.

Theorem 9.20 (Convergence of a power series)

For a power series centered at c, precisely one of the following is true.

1. *The series converges only at c.*
2. *There exists a real number $R > 0$ such that the series converges absolutely for $|x-c| < R$, and diverges for $|x-c| > R$.*
3. *The series converges absolutely for all x.*

The number R is the radius of convergence. If the series converges only at c, the radius of convergence is $R = 0$, and if the series converges for all x, the radius of convergence is $R = \infty$. The set of all values of x for which it converges is the interval of convergence of the power series.

Example 2 (Finding the radius of convergence)

Find the radius of convergence of $\sum_{n=0}^{\infty} n!x^n$.

- For $x = 0$, you obtain

$$f(0) = \sum_{n=0}^{\infty} n!0^n = 1 + 0 + 0 + \cdots = 1.$$

- For any fixed value of x such that $|x| > 0$, let $u_n = n!x^n$. Then

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = |x| \lim_{n\to\infty} (n+1) = \infty.$$

- Therefore, by the Ratio Test, the series diverges for $|x| > 0$ and converges only at its center, 0.
- So, the radius of convergence is $R = 0$. ∎

Example 3 (Finding the radius of convergence)

Find the radius of convergence of $\sum_{n=0}^{\infty} 3(x-2)^n$.

- For $x \neq 2$, let $u_n = 3(x-2)^n$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{3(x-2)^{n+1}}{3(x-2)^n}\right| = \lim_{n\to\infty}|x-2| = |x-2|.$$

- By the Ratio Test, the series converges if $|x-2| < 1$ and diverges if $|x-2| > 1$. Therefore, the radius of convergence of the series is $R = 1$. ■

Example 4 (Finding the radius of convergence)

Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

- Let $u_n = (-1)^n x^{2n+1}/(2n+1)!$. Then

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}/(2n+3)!}{(-1)^n x^{2n+1}/(2n+1)!} \right| = \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)}$$

- For any fixed value of x, this limit is 0. So, by the Ratio Test, the series converges for all x. Therefore, the radius of convergence is $R = \infty$. ∎

Differentiation and integration of power series

Theorem 9.21 (Properties of functions defined by power series)

If the function given by
$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$
has a radius of convergence of $R > 0$, then, on the interval $(c-R,\ c+R)$, f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are as follows.

1. $f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$

2.
$\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} = C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots$

The radius of convergence of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The interval of convergence, however, may differ as a result of the behavior at the endpoints.

- The interval of convergence of the series obtained by differentiating a power series may get worse but cannot get improved. However, the interval of convergence of the series obtained by integrating a power series may get improve but cannot get worse.

Example 8 (Intervals of convergence for $f(x)$, $f'(x)$, and $\int f(x)\,\mathrm{d}x$)

Consider the function given by

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots .$$

Find the interval of convergence for each of the following.
a. $\int f(x)\,\mathrm{d}x$ **b.** $f(x)$ **c.** $f'(x)$

- By Theorem 9.21, you have

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + x^3 + \cdots$$

and

$$\int f(x)\,\mathrm{d}x = C + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} = C + \frac{x^2}{1 \cdot 2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{3 \cdot 4} + \cdots .$$

- By the Ratio Test, you can show that each series has a radius of convergence of $R = 1$.
- Considering the interval $(-1, 1)$, you have the following.

a. For $\int f(x)\,\mathrm{d}x$, the series

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} \qquad \text{Interval of convergence: } [-1, 1]$$

converges for $x = \pm 1$, and its interval of convergence is $[-1, 1]$. See Figure 10.

b. For $f(x)$, the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \qquad \text{Interval of convergence: } [-1, 1)$$

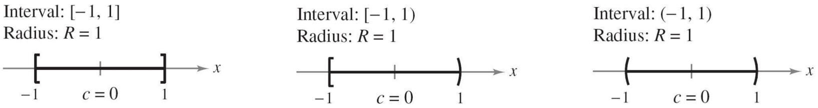
converges for $x = -1$, and diverges for $x = 1$.

- So, its interval of convergence is $[-1, 1)$. See Figure 10.

c. For $f'(x)$, the series

$$\sum_{n=1}^{\infty} x^{n-1} \qquad \text{Interval of convergence: } (-1, 1)$$

diverges for $x = \pm 1$, and its interval of convergence is $(-1, 1)$. See Figure 10. ■



(a) Interval: $[-1, 1]$ and radius: $R = 1$.

(b) Interval: $[-1, 1)$ and radius: $R = 1$.

(c) Interval: $(-1, 1)$ and radius: $R = 1$.

Figure 10: Intervals of convergence for $f(x)$, $f'(x)$, and $\int f(x)\,\mathrm{d}x$.

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Geometric power series

- Consider the function given by $f(x) = 1/(1-x)$. The form of f closely resembles the sum of a geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1.$$

- In other words, if you let $a = 1$ and $r = x$, a power series representation for $1/(1-x)$, centered at 0, is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \quad |x| < 1.$$

- Of course, this series represents $f(x) = 1/(1-x)$ only on the interval $(-1, 1)$, whereas f is defined for all $x \neq 1$, as shown in Figure 11.
- To represent f in another interval, you must develop a different series.

- For instance, to obtain the power series centered at -1, you could write
$$\frac{1}{1-x} = \frac{1}{2-(x+1)} = \frac{1/2}{1-[(x+1)/2]} = \frac{a}{1-r}$$
which implies that $a = 1/2$ and $r = (x+1)/2$.
- So, for $|x+1| < 2$, you have
$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left(\frac{x+1}{2}\right)^n \\ &= \frac{1}{2}\left[1 + \frac{(x+1)}{2} + \frac{(x+1)^2}{4} + \frac{(x+1)^3}{8} + \cdots\right], \quad |x+1| < 2\end{aligned}$$
which converges on the interval $(-3, 1)$.

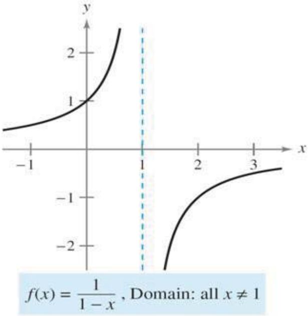

Figure 11: Definition of different ranges with function.

Example 1 (Finding a geometric power series centered at 0)

Find a power series for $f(x) = 4/(x+2)$ centered at 0.

- Writing $f(x)$ in the form $a/(1-r)$ produces

$$\frac{4}{2+x} = \frac{2}{1-(-x/2)} = \frac{a}{1-r}$$

  which implies that $a = 2$ and $r = -x/2$.

- So, the power series for $f(x)$ is

$$\frac{4}{x+2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} 2\left(-\frac{x}{2}\right)^n = 2\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots\right).$$

- This power series converges when

$$\left|-\frac{x}{2}\right| < 1$$

  which implies that the interval of convergence is $(-2, 2)$. ■

Example 2 (Finding a geometric power series centered at 1)

Find a power series for $f(x) = 1/x$, centered at 1.

- Writing $f(x)$ in the form $a/(1-r)$ produces

$$\frac{1}{x} = \frac{1}{1-(-x+1)} = \frac{a}{1-r}$$

which implies that $a = 1$ and $r = 1 - x = -(x-1)$. So, the power series for $f(x)$ is

$$\begin{aligned}\frac{1}{x} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} [-(x-1)]^n \\ &= \sum_{n=0}^{\infty} (-1)^n (x-1)^n = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots .\end{aligned}$$

- This power series converges when $|x-1| < 1$ which implies that the interval of convergence is $(0, 2)$. ■

Operations with power series Let $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$.

1. $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$

2. $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$

3. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

Example 3 (Adding two power series)

Find a power series, centered at 0, for $f(x) = (3x-1)/(x^2-1)$.

- Using partial fractions, you can write $f(x)$ as

$$\frac{3x-1}{x^2-1} = \frac{2}{x+1} + \frac{1}{x-1}.$$

- By adding the two geometric power series

$$\frac{2}{x+1} = \frac{2}{1-(-x)} = \sum_{n=0}^{\infty} 2(-1)^n x^n, \quad |x| < 1$$

and

$$\frac{1}{x-1} = \frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

you obtain the following power series.

$$\frac{3x-1}{x^2-1} = \sum_{n=0}^{\infty} [2(-1)^n - 1]x^n = 1 - 3x + x^2 - 3x^3 + x^4 - \cdots$$

- The interval of convergence for this power series is $(-1, 1)$. ■

Example 4 (Finding a power series by integration)

Find a power series for $f(x) = \ln x$, centered at 1.

- From Example 2, you know that

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n. \qquad \text{Interval of convergence: } (0,2)$$

Integrating this series produces

$$\ln x = \int \frac{1}{x}\,\mathrm{d}x + C = C + \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}.$$

- By letting $x = 1$, you can conclude that $C = 0$. Therefore,

$$\begin{aligned} \ln x &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} \\ &= \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots . \end{aligned}$$

Interval of convergence: $(0, 2]$

- Note that the series converges at $x = 2$. This is consistent with the observation in the preceding section that integration of a power series may alter the convergence at the endpoints of the interval of convergence.

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- The development of power series to represent functions is credited to the combined work of many seventeenth and eighteenth century mathematicians.
- Gregory, Newton, John and James Bernoulli, Leibniz, Euler, Lagrange, Wallis, and Fourier all contributed to this work.
- However, the two names that are most commonly associated with power series are Brook Taylor and Colin Maclaurin.

Theorem 9.22 (The form of a convergent power series)

If f is represented by a power series $f(x) = \sum a_n(x-c)^n$ for all x in an open interval I containing c, then $a_n = f^{(n)}(c)/n!$ and

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots .$$

- The coefficients of the power series in Theorem 9.22 are precisely the coefficients of the Taylor polynomials for $f(x)$ at c. For this reason, the series is called the Taylor series for $f(x)$ at c.

Definition 9.8 (Taylor and Maclaurin series)

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

is called the Taylor series for $f(x)$ at c. Moreover, if $c = 0$, then the series is the Maclaurin series for f.

Example 1 (Forming a power series)

Use the function $f(x) = \sin x$ to form the Maclaurin series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \cdots$$

and determine the interval of convergence.

- Successive differentiation of $f(x)$ yields

$$\begin{aligned} f(x) &= \sin x & f(0) &= \sin 0 = 0 \\ f'(x) &= \cos x & f'(0) &= \cos 0 = 1 \\ f''(x) &= -\sin x & f''(0) &= -\sin 0 = 0 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -\cos 0 = -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= \sin 0 = 0 \\ f^{(5)}(x) &= \cos x & f^{(5)}(0) &= \cos 0 = 1 \end{aligned}$$

and so on.

- The pattern repeats after the third derivative. So, the power series is as follows.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \cdots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = 0 + (1)x + \frac{0}{2!} x^2 + \frac{(-1)}{3!} x^3 + \frac{0}{4!} x^4 + \frac{1}{5!} x^5 + \frac{0}{6!} x^6 + \frac{(-1)}{7!} x^7 + \cdots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

- By the Ratio Test, you can conclude that this series converges for all x. ■

- You cannot conclude that the power series converges to $\sin x$ for all x. You can simply conclude that the power series converges to some function, but you are not sure what function it is.
- This is a subtle, but important, point in dealing with Taylor or Maclaurin series.
- To persuade yourself that the series

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

  might converge to a function other than f, remember that the derivatives are being evaluated at a single point.
- It can easily happen that another function will agree with the values of $f^{(n)}(x)$ when $x = c$ and disagree at other x-values.
- If you formed the power series for the function shown in Figure 12, you would obtain the same series as in Example 1.
- You know that the series converges for all x, and yet it obviously cannot converge to both $f(x)$ and $\sin x$ for all x.

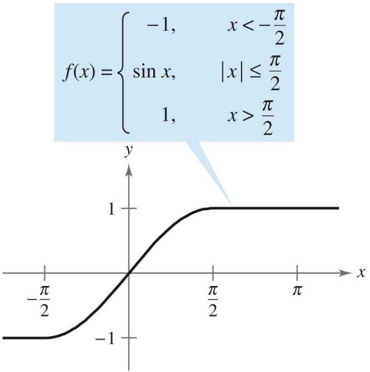


Figure 12: $f(x) \neq \sin x$ for all x but both have the same Taylor series.

- Let f have derivatives of all orders in an open interval I centered at c.
- The Taylor series for f may fail to converge for some x in I. Or, even if it is convergent, it may fail to have $f(x)$ as its sum.

- Nevertheless, Theorem 9.19 tells us that for each n,

$$f(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

Theorem 9.23 (Convergence of Taylor series)

If $\lim_{n\to\infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges and equals $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

Example 2 (A convergent Maclaurin series)

Show that the Maclaurin series for $f(x) = \sin x$ converges to $\sin x$ for all x.

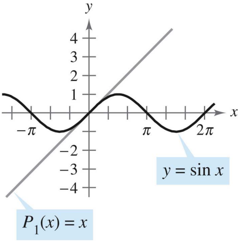
- You need to show that
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$
is true for all x.
- Because
$$f^{(n+1)}(x) = \pm \sin x \quad \text{or} \quad f^{(n+1)}(x) = \pm \cos x$$
you know that $|f^{(n+1)}(z)| \leq 1$ for every real number z.
- Therefore, for any fixed x, you can apply Taylor's Theorem (Theorem 9.19) to conclude that
$$0 \leq |R_n(x)| \leq \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

- The relative rates of convergence of exponential and factorial sequences, it follows that for a fixed x

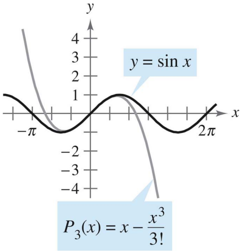
$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

- Finally, by the Squeeze Theorem, it follows that for all x, $R_n(x) \to 0$ as $n \to \infty$.
- So, by Theorem 9.23, the Maclaurin series for $\sin x$ converges to $\sin x$ for all x. ■

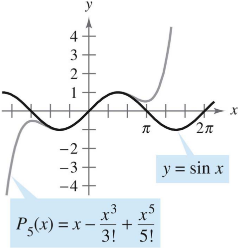
- Figure 13 visually illustrates the convergence of the Maclaurin series for $\sin x$ by comparing the graphs of the Maclaurin polynomials $P_1(x)$, $P_3(x)$, $P_5(x)$, and $P_7(x)$ with the graph of the sine function. Notice that as the degree of the polynomial increases, its graph more closely resembles that of the sine function.



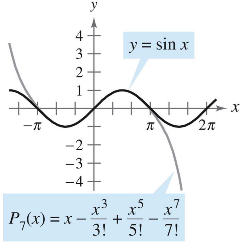
(a) $n = 1$.



(b) $n = 3$.



(c) $n = 5$.



(d) $n = 7$.

Figure 13: As n increases, the graph of P_n more closely resembles the sine function.

Guidelines for finding a Taylor series

1. Differentiate $f(x)$ several times and evaluate each derivative at c.

$$f(c), \quad f'(c), \quad f''(c), \quad f'''(c), \quad \ldots, \quad f^{(n)}(c), \quad \ldots$$

   Try to recognize a pattern in these numbers.

2. Use the sequence developed in the first step to form the Taylor coefficients $a_n = f^{(n)}(c)/n!$, and determine the interval of convergence for the resulting power series

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots .$$

3. Within this interval of convergence, determine whether or not the series converges to $f(x)$.

Example 3 (Maclaurin series for a composite function)

Find the Maclaurin series for $f(x) = \sin x^2$.

- To find the coefficients for this Maclaurin series directly, you must calculate successive derivatives of $f(x) = \sin x^2$.
- By calculating just the first two,

$$f'(x) = 2x\cos x^2 \quad \text{and} \quad f''(x) = -4x^2 \sin x^2 + 2\cos x^2$$

  you can see that this task would be quite cumbersome.
- Fortunately, there is an alternative. First consider the Maclaurin series for $\sin x$ found in Example 1.

$$g(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

- Now, because $\sin x^2 = g(x^2)$, you can substitute x^2 for x in the series for $\sin x$ to obtain

$$\sin x^2 = g(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots . \qquad \blacksquare$$

Binomial series

- Before presenting the basic list for elementary functions, you will develop one more series—for a function of the form $f(x) = (1+x)^k$.

Example 4 (Binomial series)

Find the Maclaurin series for $f(x) = (1+x)^k$ and determine its radius of convergence. Assume that k is not a positive integer.

- By successive differentiation, you have

$$
\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1 \\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\
f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1) \\
f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2) \\
&\vdots & &\vdots \\
f^{(n)}(x) &= k\cdots(k-n+1)(1+x)^{k-n} & f^{(n)} &= k(k-1)\cdots(k-n+1)
\end{aligned}
$$

- which produces the series

$$1 + kx + \frac{k(k-1)x^2}{2} + \cdots + \frac{k(k-1)\cdots(k-n+1)x^n}{n!} + \cdots .$$

- Because $a_{n+1}/a_n \to 1$, you can apply the Ratio Test to conclude that the radius of convergence is $R = 1$.
- So, the series converges to some function in the interval $(-1, 1)$. ■

Example 5 (Finding a binomial series)

Find the Maclaurin series for $f(x) = \sqrt[3]{1+x}$.

- Using the binomial series

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1+kx+\frac{k(k-1)x^2}{2!}+\frac{k(k-1)(k-2)x^3}{3!}+\cdots$$

let $k = 1/3$ and write

$$(1+x)^{1/3} = \sum_{n=0}^{\infty} \binom{1/3}{n} x^n = 1 + \frac{x}{3} - \frac{2x^2}{3^2 2!} + \frac{2 \cdot 5x^3}{3^3 3!} - \frac{2 \cdot 5 \cdot 8x^4}{3^4 4!} + \cdots$$

which converges for $-1 \le x \le 1$. ■

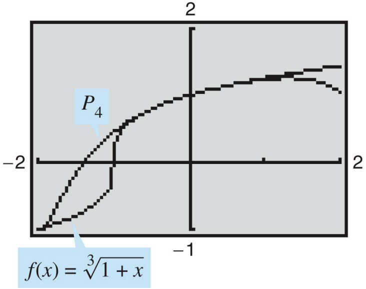


Figure 14: $f(x) = \sqrt[3]{1+x}$ and $P_4(x)$ on $[-2, 2]$.

Deriving Taylor series from a basic list

Function	Interval of convergence
$\frac{1}{x} = 1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4-\cdots+(-1)^n(x-1)^n+\cdots$	$0<x<2$
$\frac{1}{1+x} = 1-x+x^2-x^3+x^4-x^5+\cdots+(-1)^n x^n+\cdots$	$-1<x<1$
$\ln x = (x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots+\frac{(-1)^{(n-1)}(x-1)^n}{n}+\cdots$	$0<x\le 2$
$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots+\frac{x^n}{n!}+\cdots$	$-\infty<x<\infty$
$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots$	$-\infty<x<\infty$
$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots$	$-\infty<x<\infty$
$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots+\frac{(-1)^n x^{2n+1}}{2n+1}+\cdots$	$-1\le x\le 1$
$\arcsin x = x+\frac{x^3}{2\cdot 3}+\frac{1\cdot 3x^5}{2\cdot 4\cdot 5}+\frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7}+\cdots+\frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)}+\cdots$	$-1\le x\le 1$
$(1+x)^k = 1+kx+\frac{k(k-1)x^2}{2!}+\frac{k(k-1)(k-2)x^3}{3!}+\frac{k(k-1)(k-2)(k-3)x^4}{4!}+\cdots$	$-1<x<1$

The conevrgence at $x=\pm 1$ depends on the value of k.

Euler's Formula

$$e^{ix} = \cos x + i \sin x = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Example 6 (Deriving a power series from a basic list)

Find the power series for $f(x) = \cos \sqrt{x}$.

- Using the power series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

you can replace x by $\sqrt{x}$ to obtain the series

$$\cos \sqrt{x} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots .$$

- This series converges for all x in domain of $\cos\sqrt{x}$—that is, for $x \geq 0$. ■

Example 7 (Multiplication and division of power series)

Find the first three nonzero terms in each Maclaurin series $e^x \arctan x$.

- Using the Maclaurin series for e^x and $\arctan x$ in the table, you have

$$e^x \arctan x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right).$$

- Multiply these expressions and collect like terms as you would in multiplying polynomials.

$$\begin{array}{ccccccccccccc}
1 & + & x & + & \frac{1}{2}x^2 & + & \frac{1}{6}x^3 & + & \frac{1}{24}x^4 & + & \cdots & & \\
 & & x & & & - & \frac{1}{3}x^3 & & & + & \frac{1}{5}x^5 & - & \cdots \\
\hline
 & & x & + & x^2 & + & \frac{1}{2}x^3 & + & \frac{1}{6}x^4 & + & \frac{1}{24}x^5 & + & \cdots \\
 & & & & & - & \frac{1}{3}x^3 & - & \frac{1}{3}x^4 & - & \frac{1}{6}x^5 & - & \cdots \\
 & & & & & & & & & + & \frac{1}{5}x^5 & + & \cdots \\
\hline
 & & x & + & x^2 & + & \frac{1}{6}x^3 & - & \frac{1}{6}x^4 & + & \frac{3}{40}x^5 & + & \cdots
\end{array}$$

- So, $e^x \arctan x = x + x^2 + \frac{1}{6}x^3 + \cdots$. ■

Example 8 (Division of Power Series)

Find the first three nonzero terms in each Maclaurin series $\tan x$.

- Using the Maclaurin series for $\sin x$ and $\cos x$ in the table, you have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}.$$

- Divide using long division.

$$\begin{array}{r|l} & x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\ \hline 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots & x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots \\ & \underline{x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \cdots} \\ & \quad \frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots \\ & \quad \underline{\frac{1}{3}x^3 - \frac{1}{6}x^5 + \cdots} \\ & \qquad \frac{2}{15}x^5 + \cdots \end{array}$$

- So, $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$.

Example 9 (A power series for $\sin^2 x$)

Find the power series for $f(x) = \sin^2 x$.

- Consider rewriting $\sin^2 x$ as follows.

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{\cos 2x}{2}$$

- Now, use the series for $\cos x$.

$$\begin{aligned}
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \\
\cos 2x &= 1 - \frac{2^2}{2!} x^2 + \frac{2^4}{4!} x^4 - \frac{2^6}{6!} x^6 + \frac{2^8}{8!} x^8 - \cdots \\
-\frac{1}{2} \cos 2x &= -\frac{1}{2} + \frac{2}{2!} x^2 - \frac{2^3}{4!} x^4 + \frac{2^5}{6!} x^6 - \frac{2^7}{8!} x^8 + \cdots \\
\sin^2 x &= \frac{1}{2} - \frac{1}{2} \cos 2x = \frac{1}{2} - \frac{1}{2} + \frac{2}{2!} x^2 - \frac{2^3}{4!} x^4 + \frac{2^5}{6!} x^6 - \frac{2^7}{8!} x^8 + \\
&= \frac{2}{2!} x^2 - \frac{2^3}{4!} x^4 + \frac{2^5}{6!} x^6 - \frac{2^7}{8!} x^8 + \cdots
\end{aligned}$$

- This series converges for $-\infty < x < \infty$. ■

Example 10 (Power series approximation of a definite integral)

Use a power series to approximate

$$\int_0^1 e^{-x^2}\,\mathrm{d}x$$

with an error of less than 0.01.

- Replacing x with $-x^2$ in the series for e^x produces the following.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

$$\int_0^1 e^{-x^2}\,\mathrm{d}x = \left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \cdots\right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots$$

- Summing the first four terms, you have

$$\int_0^1 e^{-x^2}\,\mathrm{d}x \approx 0.74$$

which, by the Alternating Series Test, has an error of less than $1/216 \approx 0.005$. ■