Chapter 13 Functions of Several Variables

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April 20, 2022

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Functions of several variables

- The work done by a force ($W = FD$) and the volume of a right circular cylinder ($V = \pi r^2 h$) are both functions of two variables.
- The volume of a rectangular solid ($V = lwh$) is a function of three variables.
- The notation for a function of two or more variables is similar to that for a function of a single variable. Here are two examples.

$$z = f\underbrace{(x, y)}_{\text{2 variables}} = x^2 + xy \quad \text{and} \quad w = f\underbrace{(x, y, z)}_{\text{3 variables}} = x + 2y - 3z.$$

Definition 13.1 (A function of two variables)

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is called a function of x and y. The set D is the domain of f, and the corresponding set of values for $f(x, y)$ is the range of f.

- For the function given by $z = f(x, y)$, x and y are called the independent variables and z is called the dependent variable.
- As with functions of one variable, the most common way to describe a function of several variables is with an equation. In addition, the domain is the set of points for which the equation is defined.

Example 1 (Domains of functions of several variables)

Find the domain of each function.

a. $f(x,y) = \frac{\sqrt{x^2+y^2-9}}{x}$ **b.** $g(x,y,z) = \frac{x}{\sqrt{9-x^2-y^2-z^2}}$

a. The function f is defined for all points (x, y) such that $x \neq 0$ and $x^2 + y^2 \geq 9$.

- So, the domain is the set of all points lying on or outside the circle $x^2 + y^2 = 9$, except those points on the y-axis, as shown in Figure 1.

b. The function g is defined for all points (x, y, z) such that

$$x^2 + y^2 + z^2 < 9.$$

- Consequently, the domain is the set of all points (x, y, z) lying inside a sphere of radius 3 that is centered at the origin. ■

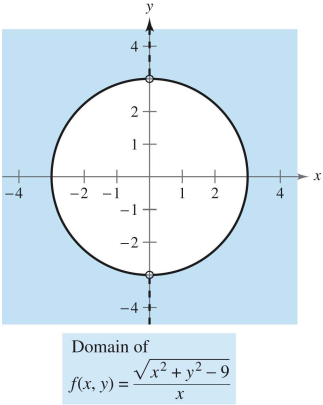


Figure 1: Domain of $f(x, y) = \frac{\sqrt{x^2+y^2-9}}{x}$.

- Functions of several variables can be combined in the same ways as functions of single variables.
- For instance, you can form the sum, difference, product, and quotient of two functions of two variables as follows.

$$
\begin{aligned}
(f \pm g)(x,y) &= f(x,y) \pm g(x,y) && \text{Sum or difference} \\
(fg)(x,y) &= f(x,y)g(x,y) && \text{Product} \\
\frac{f}{g}(x,y) &= \frac{f(x,y)}{g(x,y)}, \quad g(x,y) \neq 0 && \text{Quotient}
\end{aligned}
$$

- You cannot form the composite of two functions of several variables.
- However, if h is a function of several variables and g is a function of a single variable, you can form the composition function $(g \circ h)(x,y)$ as follows.

$$(g \circ h)(x,y) = g(h(x,y)) \qquad \text{Composition}$$

- A function that can be written as a sum of functions of the form $cx^m y^n$ (where c is a real number m and n are nonnegative integers) is called a polynomial function of two variables.
- For instance, the functions given by

$$f(x, y) = x^2 + y^2 - 2xy + x + 2 \quad \text{and} \quad g(x, y) = 3xy^2 + x - 2$$

  are polynomial functions of two variables.
- A rational function is the quotient of two polynomial functions.
- Similar terminology is used for functions of more than two variables.

The graph of a function of two variables

- The graph of a function f of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and (x, y) is in the domain of f.
- This graph can be interpreted geometrically as a surface in space.
- In Figure 2, note that the graph of $z = f(x, y)$ is a surface whose projection onto the xy-plane is the D, the domain of f.
- To each point (x, y) in D there corresponds a point (x, y, z) on the surface, and, conversely, to each point (x, y, z) on the surface there corresponds a point (x, y) in D.

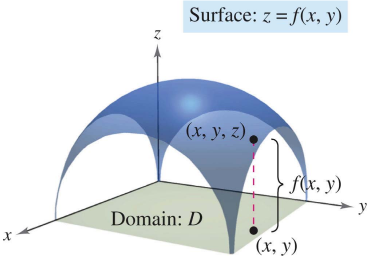


Figure 2: Surface of $z = f(x, y)$.

Example 2 (Describing the graph of a function of two variables)

Considering the function given by $f(x, y) = \sqrt{16 - 4x^2 - y^2}$. a. Find the domain and range of the function. b. Describe the graph of f.

a.

- The domain D implied by the equation of f is the set of all points (x, y) such that $16 - 4x^2 - y^2 \geq 0$.
- So, D is the set of all points lying on or inside the ellipse given by

$$\frac{x^2}{4} + \frac{y^2}{16} = 1. \qquad \text{Ellipse in the } xy\text{-plane}$$

- The range of f is all values $z = f(x, y)$ such that $0 \leq z \leq \sqrt{16}$ or

$$0 \leq z \leq 4. \qquad \text{Range of } f$$

b.

- A point (x, y, z) is on the graph of f if and only if

$$\begin{aligned} z &= \sqrt{16 - 4x^2 - y^2} \\ z^2 &= 16 - 4x^2 - y^2 \\ 4x^2 + y^2 + z^2 &= 16 \\ \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} &= 1, \quad 0 \leq z \leq 4. \end{aligned}$$

- You know that the graph of f is the upper half of an ellipsoid, as shown in Figure 3. ■

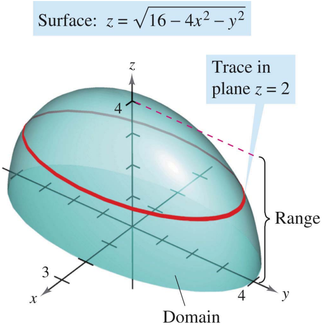
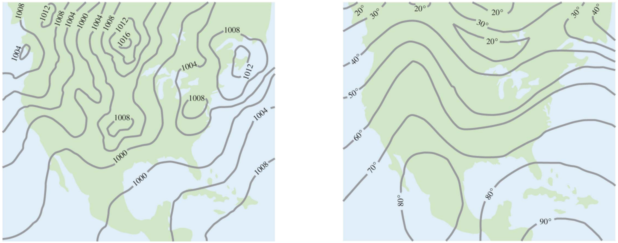


Figure 3: The graph of $f(x, y) = \sqrt{16 - 4x^2 - y^2}$ is the upper half of an ellipsoid.

Level curves

- A second way to visualize a function of two variables is to use a scalar field in which the scalar $z = f(x, y)$ is assigned to the point (x, y).
- A scalar field can be characterized by level curves (or contour lines) along which the value of $f(x, y)$ is constant.
- For instance, the weather map in Figure 4(a) shows level curves of equal pressure called isobars.
- In weather maps for which the level curves represent points of equal temperature, the level curves are called isotherms, as shown in Figure 4(b).



(a) Level curves show the lines of equal pressure (isobars) measured in milliards.

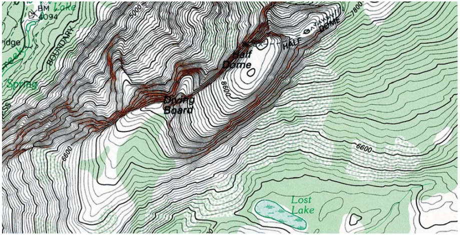
(b) Level curves show the lines of equal temperature (isotherms) measured in degrees Fahrenheit.

Figure 4: Level curves of weather map.

- Another common use of level curves is in representing electric potential fields.
- In this type of map, the level curves are called equipotential lines.
- Contour maps are commonly used to show regions on Earth's surface, with the level curves representing the height above sea level. This type of map is called a topographic map.

- For example, the mountain shown in Figure 5(a) is represented by the topographic map in Figure 5(b).
- A contour map depicts the variation of z with respect to x and y by the spacing between level curves.
- Much space between level curves indicates that z is changing slowly, whereas little space indicates a rapid change in z.
- Furthermore, to produce a good three-dimensional illusion in a contour map, it is important to choose c-values that are evenly spaced.

(a) Mountain.



(b) The topographic map of the mountain.

Figure 5: Mountain.

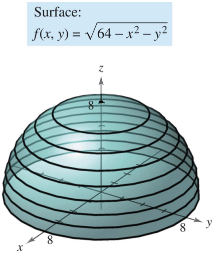
Example 3 (Sketching a contour map)

The hemisphere given by $f(x, y) = \sqrt{64 - x^2 - y^2}$ is shown in Figure 6(a). Sketch a contour map of this surface using level curves corresponding to $c = 0, 1, 2, \ldots, 8$.

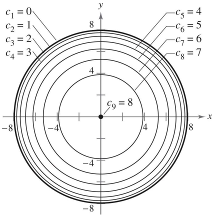
- For each value of c, the equation given by $f(x, y) = c$ is a circle (or point) in the xy-plane.
- For example, when $c_1 = 0$, the level curve is

$$x^2 + y^2 = 64 \qquad \text{Circle of radius 8}$$

  which is a circle of radius 8.
- Figure 6(b) shows the nine level curves for the hemisphere. ■



(a) Hemisphere: $f(x, y) = \sqrt{64 - x^2 - y^2}$.



(b) Contour map of $f(x, y) = \sqrt{64 - x^2 - y^2}$.

Figure 6: Contour map of hemisphere.

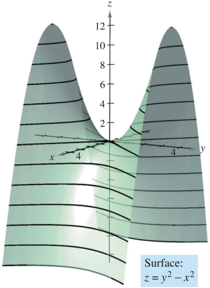
Example 4 (Sketching a contour map)

The hyperbolic paraboloid given by

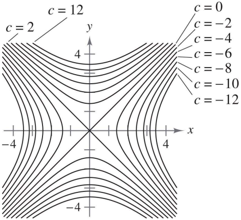
$$z = y^2 - x^2$$

is shown in Figure 7(a). Sketch a contour map of this surface.

- For each value of c, let $f(x, y) = c$ and sketch the resulting level curve in the xy-plane. For this function, each of the level curves $(c \neq 0)$ is a hyperbola whose asymptotes are the lines $y = \pm x$.
- If $c < 0$, the transverse axis is horizontal. For instance, the level curve for $c = -4$ is given by
$$\frac{x^2}{2^2} - \frac{y^2}{2^2} = 1.$$
- If $c > 0$, the transverse axis is vertical. For instance, the level curve for $c = 4$ is given by
$$\frac{y^2}{2^2} - \frac{x^2}{2^2} = 1.$$
- If $c = 0$, the level curve is the degenerate conic representing the intersecting asymptotes, as shown in Figure 7(b). ■



(a) Hyperbolic paraboloid.



(b) Hyperbolic level curves (at increments of 2).

Figure 7: Level curves of hyperbolic paraboloid.

Level surfaces

- The concept of a level curve can be extended by one dimension to define a level surface.
- If f is a function of three variables and c is a constant, the graph of the equation $f(x, y, z) = c$ is a level surface of the function f, as shown in Figure 8.

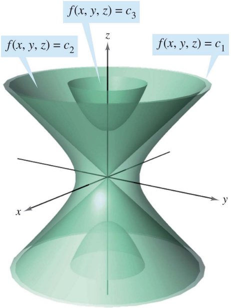


Figure 8: Level surfaces of $f(x, y, z) = c$.

Example 6 (Level surfaces)

Describe the level surfaces of the function

$$f(x, y, z) = 4x^2 + y^2 + z^2.$$

- Each level surface has an equation of the form

$$4x^2 + y^2 + z^2 = c. \qquad \text{Equation of level surface}$$

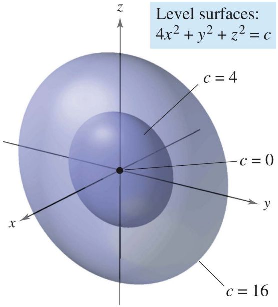
- So, the level surfaces are ellipsoids (whose cross sections parallel to the yz-plane are circles).
- As c increases, the radii of the circular cross sections increase according to the square root of c.
- For example, the level surfaces corresponding to the values $c = 0$, $c = 4$, $c = 16$ and are as follows.

$$4x^2 + y^2 + z^2 = 0 \qquad \text{Level surface for } c = 0 \text{ (single point)}$$

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{4} = 1 \qquad \text{Level surface for } c = 4 \text{ (ellipsoid)}$$

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1 \qquad \text{Level surface for } c = 16 \text{ (ellipsoid)}$$

- These level surfaces are shown below. ■
- If the function represents the temperature at the point (x, y, z), the level surfaces shown below would be called isothermal surfaces.



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Neighborhoods in the plane

Using the formula for the distance between two points (x, y) and (x_0, y_0) in the plane, you can define the δ-neighborhood about (x_0, y_0) to be the disk centered at (x_0, y_0) with radius $\delta > 0$

$$\left\{(x, y) : \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\right\} \qquad \text{Open disk}$$

as shown in Figure 9.

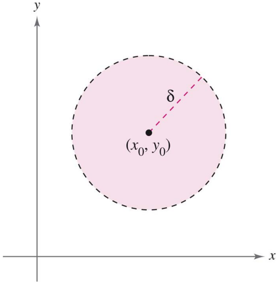


Figure 9: An open disk $\left\{(x, y) : \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\right\}$.

- When this formula contains the less than inequality sign, $<$, the disk is called open, and when it contains the less than or equal to inequality sign, $\leq$, the disk is called closed. This corresponds to the use of $<$ and $\leq$ to define open and closed intervals.
- A point (x_0, y_0) in a plane region R is an interior point of R if there exists a δ-neighborhood about (x_0, y_0) that lies entirely in R, as shown in Figure 10.

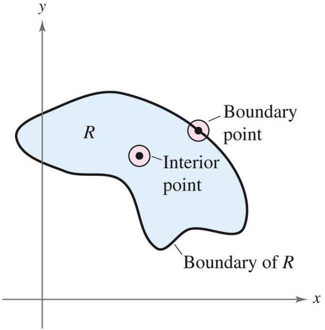


Figure 10: The boundary and interior points of a region R.

- If every point in R is an interior point, then R is an open region. A point (x_0, y_0) is a boundary point of R if every open disk centered at (x_0, y_0) contains points inside R and points outside R.
- By definition, a region must contain its interior points, but it need not contain its boundary points. If a region contains all its boundary points, the region is closed region. A region that contains some but not all of its boundary points is neither open nor closed.

Limit of a function of two variables

Definition 13.2 (Limit of a function of two variables)

Let f be a function of two variables defined, except possible at (x_0, y_0), on an open disk centered at (x_0, y_0), and let L be a real number. Then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for each ε there corresponds a $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$

- Graphically, this definition of a limit implies that for any point $(x, y) \neq (x_0, y_0)$ in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$, as shown in Figure 11.

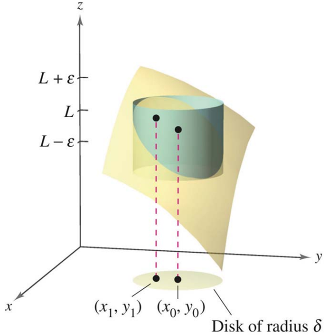


Figure 11: For any (x, y) in the disk of radius δ, the value $f(x, y)$ lies between $L+\varepsilon$ and $L-\varepsilon$.

- The definition of the limit of a function of two variables is similar to the definition of the limit of a function of a single variable, yet there is a critical difference.
- To determine whether a function of a single variable has a limit, you need only test the approach from two directions—from the right and from the left.

- If the function approaches the same limit from the right and from the left, you can conclude that the limit exists.
- However, for a function of two variables, the statement

$$(x, y) \to (x_0, y_0)$$

means that the point (x, y) is allowed to approach (x_0, y_0) from any direction.
- If the value of

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y)$$

is not the same for all possible approaches, or paths, to (x_0, y_0), the limit does not exist.

Example 1 (Verifying a limit by the definition)

Show that $\lim_{(x,y)\to(a,b)} x = a$.

- Let $f(x, y) = x$ and $L = a$.
- You need to show that for each $\varepsilon > 0$, there exists a δ-neighborhood about (a, b) such that

$$|f(x, y) - L| = |x - a| < \varepsilon$$

whenever $(x, y) \neq (a, b)$ lies in the neighborhood.

- You can first observe that from

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

it follows that

$$|f(x, y) - L| = |x - a| = \sqrt{(x-a)^2} \leq \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

- So, you can choose $\delta = \varepsilon$, and the limit is verified. ■
- Limits of functions of several variables have the same properties regarding sums, differences, products, and quotients as do limits of functions of single variables.

Example 2 (Verifying a limit)

Evaluate $\lim_{(x,y)\to(1,2)} \frac{5x^2y}{x^2+y^2}$.

- By using the properties of limits of products and sums, you obtain

$$\lim_{(x,y)\to(1,2)} 5x^2y = 5(1^2)(2) = 10$$

and

$$\lim_{(x,y)\to(1,2)} (x^2+y^2) = (1^2+2^2) = 5.$$

- Because the limit of a quotient is equal to the quotient of the limits (and the denominator is not 0), you have

$$\lim_{(x,y)\to(1,2)} \frac{5x^2y}{x^2+y^2} = \frac{10}{5} = 2. \qquad \blacksquare$$

Example 3 (Finding a limit)

Evaluate $\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}$.

- The limits of the numerator and of the denominator are both 0, and so you cannot determine the existence of a limit by taking the limits of the numerator and denominator separately and then dividing.
- However, from the graph of f in Figure 12, it seems reasonable that the limit might be 0. So, you can try applying the definition to $L = 0$.
- First, note that

$$|y| \leq \sqrt{x^2+y^2} \quad \text{and} \quad \frac{x^2}{x^2+y^2} \leq 1.$$

- Then, in a δ-neighborhood about $(0,0)$, you have $0 < \sqrt{x^2+y^2} < \delta$, and it follows that, for $(x,y) \neq (0,0)$,

$$|f(x,y) - 0| = \left|\frac{5x^2y}{x^2+y^2}\right| = 5|y|\left(\frac{x^2}{x^2+y^2}\right) \leq 5|y| \leq 5\sqrt{x^2+y^2} \leq 5\delta.$$

- So, you can choose $\delta = \epsilon/5$ and conclude that

$$\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2} = 0.$$

$\blacksquare$

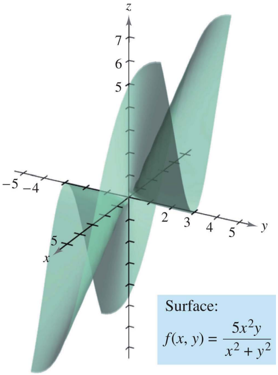


Figure 12: $\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2} = 0.$

- For some functions, it is easy to recognize that a limit does not exist.
- For instance, it is clear that the limit $\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2}$ does not exist because the values of $f(x, y)$ increase without bound as (x, y) approaches $(0, 0)$ along any path (see Figure 13).

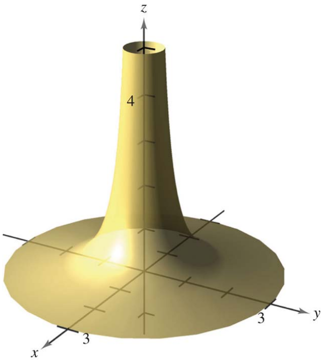


Figure 13: $\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2}$ does not exist.

Example 4 (A limit that does not exist)

Show that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$$

- Along the x-axis, every point is of the form $(x, 0)$, and the limit along this approach is

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^2 - 0^2}{x^2 + 0^2}\right)^2 = \lim_{(x,y)\to(0,0)} 1^2 = 1. \qquad \text{Limit along } x\text{-axis}$$

- However, if (x, y) approaches $(0, 0)$ along the line $y = x$, you obtain

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^2 - x^2}{x^2 + x^2}\right)^2 = \lim_{(x,y)\to(0,0)} 0^2 = 0. \qquad \text{Limit along line } y = x$$

- So, f does not have a limit as $(x, y) \to (0, 0)$. ■

Continuity of a function of two variables

- The limit of $f(x, y) = 5x^2y/(x^2 + y^2)$ as $(x, y) \to (1, 2)$ can be evaluated by direct substitution. That is, the limit is $f(1, 2) = 2$.
- In such cases the function f is said to be continuous at the point $(1, 2)$.

Definition 13.3 (Continuity of a function of two variables)

A function f of two variables is continuous at a point (x_0, y_0) in an open region R if $f(x_0, y_0)$ is equal to the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0). That is,

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0).$$

The function f is continuous in the open region R if it is continuous at every point in R.

- The function $f(x,y) = \frac{5x^2y}{x^2+y^2}$ is not continuous at $(0,0)$. However, because the limit at this point exists, you can remove the discontinuity by defining f at $(0,0)$ as being equal to its limit there. Such a discontinuity is call removable.
- The function $f(x,y) = \left[\frac{x^2-y^2}{x^2+y^2}\right]^2$ is not continuous at $(0,0)$, and this discontinuity is nonremovable.

Theorem 13.1 (Continuity of a function of two variables)

If k is a real number and f and g are continuous at (x_0, y_0), then the following function are continuous at (x_0, y_0).

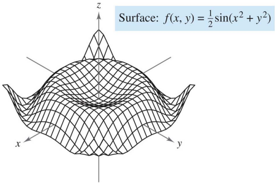
1. *Scalar multiple: kf*

2. *Product: fg*

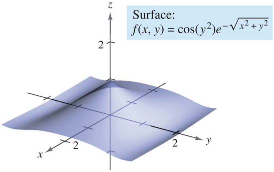
3. *Sum and difference: $f \pm g$*

4. *Quotient: f/g, if $g(x_0, y_0) \neq 0$.*

- Theorem 13.1 establishes the continuity of polynomial and rational functions at every point in their domains. Furthermore, the continuity of other types of functions can be extended naturally from one to two variables.
- For instance, the functions whose graphs are shown in Figures 14(a) and 14(b) are continuous at every point in the plane.



(a) The function $f(x, y) = \frac{1}{2}\sin(x^2 + y^2)$ is continuous at every point in the plane.



(b) The function $f(x, y) = \cos(y^2)e^{-\sqrt{x^2+y^2}}$ is continuous at every point in the plane.

Figure 14: Surfaces about continuity.

Theorem 13.2 (Continuity of a composite function)

If h is continuous at (x_0, y_0) and g is continuous at $h(x_0, y_0)$, then the composite function given by $(g \circ h)(x, y) = g(h(x, y))$ is continuous at (x_0, y_0). That is,

$$\lim_{(x,y)\to(x_0,y_0)} g(h(x, y)) = g(h(x_0, y_0)).$$

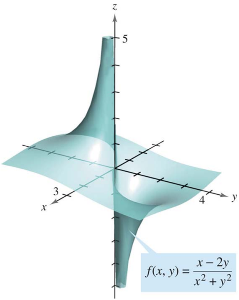
Example 5 (Testing for continuity)

Discuss the continuity of each function.
a. $f(x, y) = \frac{x-2y}{x^2+y^2}$ **b.** $g(x, y) = \frac{2}{y-x^2}$.

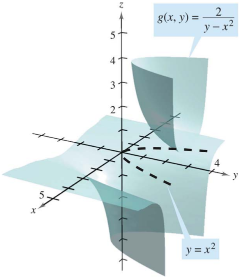
a. Because a rational function is continuous at every point in its domain, you can conclude that f is continuous at each point in the xy-plane except at $(0, 0)$, as shown in next slide.

b. The function given by $g(x,y) = 2/(y-x^2)$ is continuous except at the points at which the denominator is 0, $y - x^2 = 0$.

- So, you can conclude that the function is continuous at all points except those lying on the parabola $y = x^2$.
- Inside this parabola, you have $y > x^2$, and the surface represented by the function lies above the xy-plane, as shown below.
- Outside parabola, $y < x^2$, and the surface lies below the xy-plane. ■



(a) The function $f(x,y) = \frac{x-2y}{x^2+y^2}$ is not continuous at $(0,0)$.



(b) The function $g(x,y) = \frac{2}{y-x^2}$ is not continuous on the parabola $y = x^2$.

Continuity of a function of three variables

- The definitions of limits and continuity can be extended to functions of three variables by considering points (x, y, z) within the open sphere

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2. \qquad \text{Open sphere}$$

- The radius of this sphere is δ, and the sphere is centered at (x_0, y_0, z_0), as shown in Figure 16.

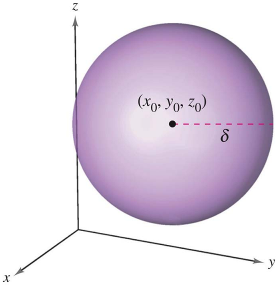


Figure 16: Open sphere $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2$ in space.

- A point (x_0, y_0, z_0) in a region R in space is an interior point of R if there exists a δ-sphere about (x_0, y_0, z_0) that lies entirely in R. If every point in R is an interior point, then R is called open.

Definition 13.4 (Continuity of a function of three variables)

A function f of three variables is continuous at a point (x_0, y_0, z_0) in an open region R if $f(x_0, y_0, z_0)$ is defined and is equal to the limit of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0). That is,

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0).$$

The function f is continuous in the open region R if it is continuous at every point in R.

Example 6 (Testing continuity of a function of three variables)

Discuss the continuity of $f(x, y, z) = \frac{1}{x^2+y^2-z}$.

The function

$$f(x, y, z) = \frac{1}{x^2 + y^2 - z}$$

is continuous at each point in space except at the points on the paraboloid given by $z = x^2 + y^2$. ■

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Partial derivatives of a function of two variables

- You can determine the rate of change of f with respect to one of its several independent variables. This process is called partial differentiation, and the result is referred to as the partial derivative of f with respect to the chosen independent variable.

Definition 13.5 (Partial derivatives of a function of two variables)

If $z = f(x, y)$, then the first partial derivatives of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

Partial derivative with respect to x and

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Partial derivative with respect to y , provided the limits exist.

- This definition indicates that if $z = f(x, y)$, then to find f_x you consider y constant and differentiate with respect to x.
- Similarly, to find f_y, you consider x constant and differentiate with respect to y.

Example 1 (Finding partial derivatives)

Find the partial derivatives f_x and f_y for the function
a. $f(x, y) = 3x - x^2y^2 + 2x^3y$. **b.** $f(x, y) = (\ln x)(\sin x^2y)$.

a. Considering y to be constant and differentiating with respect to x produces

$$f(x, y) = 3x - x^2y^2 + 2x^3y \qquad f_x(x, y) = 3 - 2xy^2 + 6x^2y.$$

- Considering x to be constant and differentiating with respect to y produces

$$f(x, y) = 3x - x^2y^2 + 2x^3y \qquad f_y(x, y) = -2x^2y + 2x^3.$$

b. Considering y to be constant and differentiating with respect to x produces

$$f(x, y) = (\ln x)(\sin x^2 y) \quad f_x(x, y) = (\ln x)(\cos x^2 y)(2xy) + \frac{\sin x^2 y}{x}.$$

- Considering x to be constant and differentiating with respect to y produces

$$f(x, y) = (\ln x)(\sin x^2 y) \qquad f_y(x, y) = (\ln x)(\cos x^2 y)(x^2). \qquad \blacksquare$$

(Notation for first partial derivatives)
For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}$$

and

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}.$$

The first partials evaluated at the point (a, b) are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = f_x(a, b) \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(a,b)} = f_y(a, b).$$

Example 2 (Finding and evaluating partial derivatives)

For $f(x, y) = xe^{x^2 y}$, find f_x and f_y, and evaluate each at the point $(1, \ln 2)$.

- Because
$$f_x(x, y) = xe^{x^2 y}(2xy) + e^{x^2 y}$$
the partial derivative of f with respect to x at $(1, \ln 2)$ is
$$f_x(1, \ln 2) = e^{\ln 2}(2\ln 2) + e^{\ln 2} = 4\ln 2 + 2.$$
- Because
$$f_y(x, y) = xe^{x^2 y}(x^2) = x^3 e^{x^2 y}$$
the partial derivative of f with respect to y at $(1, \ln 2)$ is
$$f_y(1, \ln 2) = e^{\ln 2} = 2. \qquad \blacksquare$$

- The partial derivatives of a function of two variables, $z = f(x, y)$, have a useful geometric interpretation. If $y = y_0$, then $z = f(x, y_0)$ represents the curve formed by intersecting the surface $z = f(x, y)$ with the plane $y = y_0$, as shown in Figure 17(a).
- Therefore,

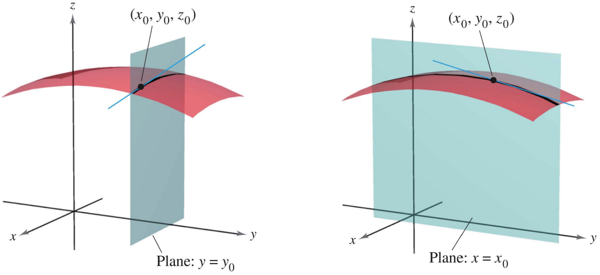
$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

  represents the slope of this curve at the point $(x_0, y_0, f(x_0, y_0))$.
- Note that both the curve and the tangent line lie in the plane $y = y_0$.
- Similarly,

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

  represents the slope of the curve given by the intersection of $z = f(x, y)$ and the plane $x = x_0$ at $(x_0, y_0, f(x_0, y_0))$, as shown in Figure 17(b).

- Informally, the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point (x_0, y_0, z_0) denote the slopes of the surface in the x- and y-directions , respectively.



(a) $\frac{\partial f}{\partial x}$ = slope in x-direction.

(b) $\frac{\partial f}{\partial y}$ = slope in y-direction.

Figure 17: Partial derivatives of a function of two variables.

Example 3 (Finding the slopes of a surface in the x- and y-directions)

Find the slopes in the x-direction and in the y-direction of the surface given by

$$f(x, y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$$

at the point $\left(\frac{1}{2}, 1, 2\right)$.

- The partial derivatives of f with respect to x and y are

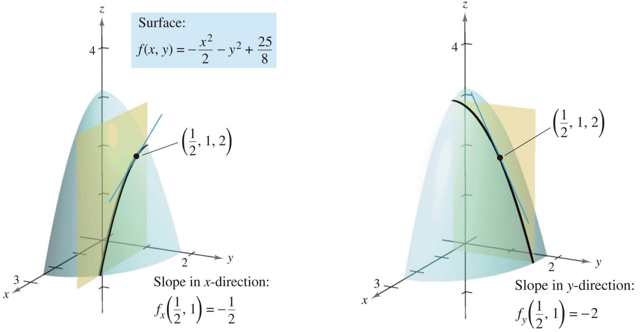
$$f_x(x, y) = -x \quad \text{and} \quad f_y(x, y) = -2y. \qquad \text{Partial derivatives}$$

- So, in the x-direction, the slope is

$$f_x\left(\frac{1}{2}, 1\right) = -\frac{1}{2} \qquad \text{Figure 18(a)}$$

and in the y-direction, the slope is

$$f_y\left(\frac{1}{2}, 1\right) = -2. \qquad \text{Figure 18(b)}$$



(a) Slope in x-direction: $f_x\left(\frac{1}{2}, 1\right) = -\frac{1}{2}$.

(b) Slope in y-direction: $f_y\left(\frac{1}{2}, 1\right) = -2$.

Figure 18: Partial derivatives of $f(x, y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$ at $\left(\frac{1}{2}, 1, 2\right)$.

Example 4 (Finding the slopes of a surface in the x- and y-directions)

Find the slopes of the surface given by

$$f(x,y) = 1 - (x-1)^2 - (y-2)^2$$

at the point $(1,2,1)$ in the x-direction and in the y-direction.

- The partial derivatives of f with respect to x and y are

$$f_x(x,y) = -2(x-1) \quad \text{and} \quad f_y(x,y) = -2(y-2).$$

- So, at the point $(1,2,1)$, the slopes in the x- and y-directions are

$$f_x(1,2) = -2(1-1) = 0 \quad \text{and} \quad f_y(1,2) = -2(2-2) = 0.$$

as shown in Figure 19 ■

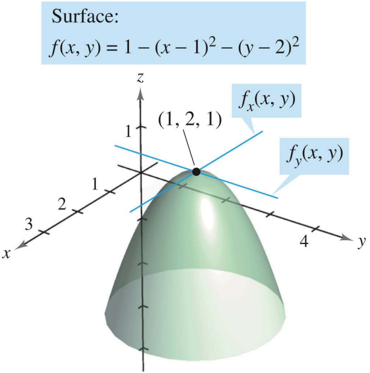


Figure 19: Finding the slopes of a surface in the x- and y-directions.

No matter how many variables are involved, partial derivatives can be interpreted as rates of change.

Example 5 (Using partial derivatives to find rates of change)

The area of a parallelogram with adjacent sides a and b and included angle θ is given by $A = ab\sin\theta$, as shown in Figure 20.

a. Find the rate of change of A with respect to a for $a = 10$, $b = 20$, and $\theta = \frac{\pi}{6}$.

b. Find the rate of change of A with respect to θ for $a = 10$, $b = 20$, and $\theta = \frac{\pi}{6}$.

a. To find the rate of change of the area with respect to a, hold b and θ constant and differentiate with respect to a to obtain

$$\frac{\partial A}{\partial a} = b\sin\theta \qquad \left.\frac{\partial A}{\partial a}\right|_{b=20,\theta=\frac{\pi}{6}} = 20\sin\frac{\pi}{6} = 10.$$

b. To find the rate of change of the area with respect to θ, hold a and b constant and differentiate with respect to θ to obtain

$$\frac{\partial A}{\partial \theta} = ab\cos\theta \qquad \left.\frac{\partial A}{\partial \theta}\right|_{a=10,b=20,\theta=\frac{\pi}{6}} = 200\cos\frac{\pi}{6} = 100\sqrt{3}. \quad \blacksquare$$

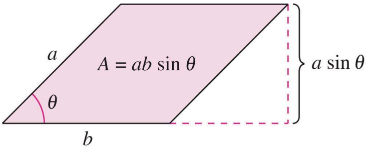


Figure 20: The area of the parallelogram is $ab\sin\theta$.

Partial derivatives of a function of three or more variables

- The concept of a partial derivative can be extended naturally to functions of three or more variables. For instance, if $w = f(x, y, z)$, there are three partial derivatives, each of which is formed by holding two of the variables constant.
- That is, to define the partial derivative of w with respect to x, consider y and z to be constant and differentiate with respect to x.
- A similar process is used to find the derivatives of w with respect to y and with respect to z.

$$\frac{\partial w}{\partial x} = f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$
$$\frac{\partial w}{\partial y} = f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$
$$\frac{\partial w}{\partial z} = f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

- In general, if $w = f(x_1, x_2, \ldots, x_n)$, there are n partial derivatives denoted by

$$\frac{\partial w}{\partial x_k} = f_{x_k}(x_1, x_2, \ldots, x_n), \quad k = 1,\ 2,\ \ldots,\ n.$$

- To find the partial derivative with respect to one of the variables, hold the other variables constant and differentiate with respect to the given variable.

Example 6 (Finding partial derivatives)

a. To find the partial derivative of $f(x, y, z) = xy + yz^2 + xz$ with respect to z, consider x and y to be constant and obtain

$$\frac{\partial}{\partial z}[xy + yz^2 + xz] = 2yz + x.$$

b. To find the partial derivative of $f(x, y, z) = z\sin(xy^2 + 2z)$ with respect to z, consider x and y to be constant. Then, using the Product Rule, you obtain

$$\begin{aligned}\frac{\partial}{\partial z}[z\sin(xy^2+2z)] &= (z)\frac{\partial}{\partial z}[\sin(xy^2+2z)] + \sin(xy^2+2z)\frac{\partial}{\partial z}[z] \\ &= (z)[\cos(xy^2+2z)](2) + \sin(xy^2+2z) \\ &= 2z\cos(xy^2+2z) + \sin(xy^2+2z).\end{aligned}$$

c. To find the partial derivative of $f(x, y, z, w) = (x + y + z)/w$ with respect to w, consider x, y, and z to be constant and obtain

$$\frac{\partial}{\partial w}\left[\frac{x+y+z}{w}\right] = -\frac{x+y+z}{w^2}.$$

■

Higher-order partial derivatives

- The function $z = f(x, y)$ has the following second partial derivatives.
  1. Differentiate twice with respect to x:
  $$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}.$$
  2. Differentiate twice with respect to y:
  $$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$
  3. Differentiate first with respect to x and then with respect to y:
  $$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.$$
  4. Differentiate first with respect to y and then with respect to x:
  $$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$
- The third and fourth cases are called mixed partial derivatives.

Example 7 (Finding second partial derivatives)

Find the second partial derivatives of $f(x, y) = 3xy^2 - 2y + 5x^2y^2$, and determine the value of $f_{xy}(-1, 2)$.

- Begin by finding the first partial derivatives with respect to x and y.

$$f_x(x, y) = 3y^2 + 10xy^2 \quad \text{and} \quad f_y(x, y) = 6xy - 2 + 10x^2y$$

- Then, differentiate each of these with respect to x and y.

$$\begin{aligned} f_{xx}(x, y) &= 10y^2 & \text{and} \qquad f_{yy}(x, y) &= 6x + 10x^2 \\ f_{xy}(x, y) &= 6y + 20xy & \text{and} \qquad f_{yx}(x, y) &= 6y + 20xy \end{aligned}$$

At $(-1, 2)$, the value of f_{xy} is $f_{xy}(-1, 2) = 12 - 40 = -28$. ■

Theorem 13.3 (Equality of mixed partial derivatives)

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then, for every (x, y) in R,

$$f_{xy}(x, y) = f_{yx}(x, y).$$

Example 8 (Finding higher-order partial derivatives)

Show that $f_{xz} = f_{zx}$ and $f_{xzz} = f_{zxz} = f_{zzx}$ for the function given by $f(x, y, z) = ye^x + x \ln z$.

- First partials:

$$f_x(x, y, z) = ye^x + \ln z, \quad f_z(x, y, z) = \frac{x}{z}$$

- Second partials (note that the first two are equal):

$$f_{xz}(x, y, z) = \frac{1}{z}, \quad f_{zx}(x, y, z) = \frac{1}{z}, \quad f_{zz}(x, y, z) = -\frac{x}{z^2}$$

- Third partials (note that all three are equal):

$$f_{xzz}(x, y, z) = -\frac{1}{z^2}, \quad f_{zxz}(x, y, z) = -\frac{1}{z^2}, \quad f_{zzx}(x, y, z) = -\frac{1}{z^2}$$

■

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Increments and differentials

- For $y = f(x)$, the differential of y was defined as $\mathrm{d}y = f'(x)\,\mathrm{d}x$.
- Similar terminology is used for a function of two variables, $z = f(x, y)$. That is, Δx and Δy are the increments of x and y, and the increment of z is given by

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y). \qquad \text{Increment of } z$$

Definition 13.6 (Total differential)

If $z = f(x, y)$ and Δx and Δy are increments of x and y, then the differentials of the independent variables x and y are

$$\mathrm{d}x = \Delta x \quad \text{and} \quad \mathrm{d}y = \Delta y$$

and the total differential of the dependent variable z is

$$\mathrm{d}z = \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y = f_x(x, y)\,\mathrm{d}x + f_y(x, y)\,\mathrm{d}y.$$

- This definition can be extended to a function of more variables.
- For instance, if $w = f(x, y, z, u)$, then $\mathrm{d}x = \Delta x$, $\mathrm{d}y = \Delta y$, $\mathrm{d}z = \Delta z$, $\mathrm{d}u = \Delta u$, and the total differential of w is

$$\mathrm{d}w = \frac{\partial w}{\partial x}\,\mathrm{d}x + \frac{\partial w}{\partial y}\,\mathrm{d}y + \frac{\partial w}{\partial z}\,\mathrm{d}z + \frac{\partial w}{\partial u}\,\mathrm{d}u.$$

Example 1 (Finding the total differential)

Find the total differential for each function.
a. $z = 2x\sin y - 3x^2y^2$ **b.** $w = x^2 + y^2 + z^2$

a. The total differential $\mathrm{d}z$ for $z = 2x\sin y - 3x^2y^2$ is

$$\mathrm{d}z = \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y = (2\sin y - 6xy^2)\,\mathrm{d}x + (2x\cos y - 6x^2y)\,\mathrm{d}y.$$

b. The total differential $\mathrm{d}w$ for $w = x^2 + y^2 + z^2$ is

$$\mathrm{d}w = \frac{\partial w}{\partial x}\,\mathrm{d}x + \frac{\partial w}{\partial y}\,\mathrm{d}y + \frac{\partial w}{\partial z}\,\mathrm{d}z = 2x\,\mathrm{d}x + 2y\,\mathrm{d}y + 2z\,\mathrm{d}z. \quad \blacksquare$$

Differentiability

- For a differentiable function given by $y = f(x)$, you can use the differential $\mathrm{d}y = f'(x)\,\mathrm{d}x$ as an approximation (for small Δx) to the value $\Delta y = f(x + \Delta x) - f(x)$.
- When a similar approximation is possible for a function of two variables, the function is said to be differentiable.

Definition 13.7 (Differentiability)

A function f given by $z = f(x, y)$ is differentiable at (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where both ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. The function f is differentiable in a region R if it is differentiable at each point in R.

Example 2 (Showing that a function is differentiable)

Show that the function given by

$$f(x, y) = x^2 + 3y$$

is differentiable at every point in the plane.

- Letting $z = f(x, y)$, the increment of z at an arbitrary point (x, y) in the plane is

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= (x^2 + 2x\Delta x + \Delta x^2) + 3(y + \Delta y) - (x^2 + 3y)\\ &= 2x\Delta x + \Delta x^2 + 3\Delta y = 2x(\Delta x) + 3(\Delta y) + \Delta x(\Delta x) + 0(\Delta y)\\ &= f_x(x, y)\Delta x + f_y(x, y)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y\end{aligned}$$

  where $\epsilon_1 = \Delta x$ and $\epsilon_2 = 0$.
- Because $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$, it follows that f is differentiable at every point in the plane. ■

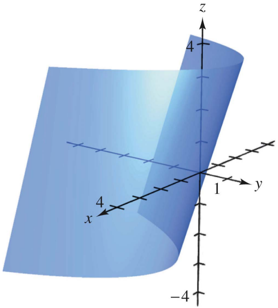

Figure 21: $f(x, y) = x^2 + 3y$ is differentiable at every point in the plane.

Theorem 13.4 (Sufficient condition for differentiability)

If f is a function of x and y, where f_x and f_y are continuous in an open region R, then f is differentiable on R.

Approximation by differentials

- Theorem 13.4 tells you that you can choose $(x + \Delta x, y + \Delta y)$ close enough to (x, y) to make $\epsilon_1 \Delta x$ and $\epsilon_2 \Delta y$ insignificant. In other words, for small Δx and Δy, you can use the approximation $\Delta z \approx \mathrm{d}z$.

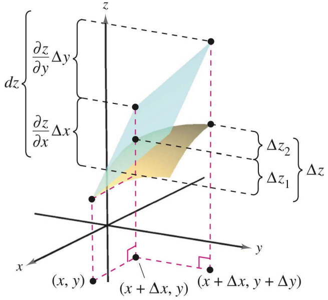

Figure 22: The exact change in z is Δz. This change can be approximated by the differential $\mathrm{d}z$.

- The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ can be interpreted as the slopes of the surface in the x- and y-directions.
- This means that

$$\mathrm{d}z = \frac{\partial z}{\partial x}\,\Delta x + \frac{\partial z}{\partial y}\,\Delta y$$

  represents the change in height of a plane that is tangent to the surface at the point $(x, y, f(x, y))$.
- Because a plane in space is represented by a linear equation in the variables x, y, and z, the approximation of Δz by $\mathrm{d}z$ is called a linear approximation.

Example 3 (Using a differential as an approximation)

Use the differential $\mathrm{d}z$ to approximate the change in $z = \sqrt{4 - x^2 - y^2}$ as (x, y) moves from the point $(1, 1)$ to the point $(1.01, 0.97)$. Compare this approximation with the exact change in z.

- Letting $(x, y) = (1, 1)$ and $(x + \Delta x, y + \Delta y) = (1.01, 0.97)$ produces $\mathrm{d}x = \Delta x = 0.01$ and $\mathrm{d}y = \Delta y = -0.03$.
- So, the change in z can be approximated by

$$\Delta z \approx \mathrm{d}z = \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y = \frac{-x}{\sqrt{4 - x^2 - y^2}}\,\Delta x + \frac{-y}{\sqrt{4 - x^2 - y^2}}\,\Delta y.$$

- When $x = 1$ and $y = 1$, you have

$$\Delta z \approx -\frac{1}{\sqrt{2}}(0.01) - \frac{1}{\sqrt{2}}(-0.03) = \frac{0.02}{\sqrt{2}} \approx 0.0141.$$

- In Figure 23, you can see that the exact change corresponds to the difference in the heights of two points on the surface of a hemisphere.
- This difference is given by

$$\begin{aligned}\Delta z &= f(1.01, 0.97) - f(1, 1)\\ &= \sqrt{4^2 - (1.01)^2 - (0.97)^2} - \sqrt{4^2 - 1^2 - 1^2} \approx 0.0137. \qquad \blacksquare\end{aligned}$$

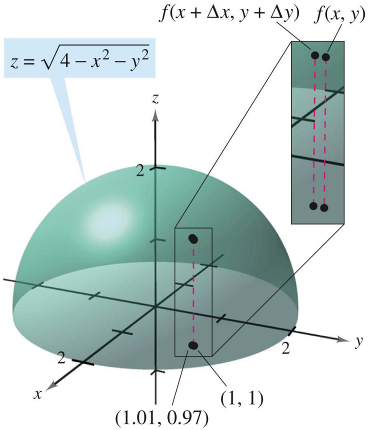

Figure 23: As (x, y) moves from $(1, 1)$ to the point $(1.01, 0.97)$, the value of $f(x, y)$ changes by about 0.0137.

- A function of three variables $w = f(x, y, z)$ is called differentiable at (x, y, z) provided that

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

can be written in the form

$$\Delta w = f_x \, \Delta x + f_y \, \Delta y + f_z \, \Delta z + \epsilon_1 \, \Delta x + \epsilon_2 \, \Delta y + \epsilon_3 \, \Delta z$$

where ϵ_1, ϵ_2, and $\epsilon_3 \to 0$ as $(\Delta x, \Delta y, \Delta z) \to (0, 0, 0)$.
- With this definition of differentiability, Theorem 13.4 has the following extension for functions of three variables:

Sufficient condition for differentiability: If f is a function of x, y, and z, where f, f_x, f_y, and f_z are continuous in an open region R, then f is differentiable on R.

Theorem 13.5 (Differentiability implies continuity)

If a function of x and y is differentiable at (x_0, y_0), then it is continuous at (x_0, y_0).

- Let f be differentiable at (x_0, y_0), where $z = f(x, y)$. Then

$$\Delta z = [f_x(x_0, y_0) + \varepsilon_1]\Delta x + [f_y(x_0, y_0) + \varepsilon_2]\Delta y$$

  where both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.
- However, by definition, you know that Δz is given by

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

- Letting $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$ produce

$$\begin{aligned} f(x, y) - f(x_0, y_0) &= [f_x(x_0, y_0) + \varepsilon_1]\Delta x + [f_y(x_0, y_0) + \varepsilon_2]\Delta y \\ &= [f_x(x_0, y_0) + \varepsilon_1](x - x_0) + [f_y(x_0, y_0) + \varepsilon_2](y - y_0). \end{aligned}$$

- Taking the limit as $(x, y) \to (x_0, y_0)$, you have

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

which means that f is continuous at (x_0, y_0). $\square$

The existence of f_x and f_y is not sufficient to guarantee differentiability, as illustrated in the next example.

Example 5 (A function that is not differentiable)

Show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist, but that f is not differentiable at $(0, 0)$ where f is defined as

$$f(x, y) = \begin{cases} \frac{-3xy}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}.$$

- You can show that f is not differentiable at $(0,0)$ by showing that it is not continuous at this point.
- To see that f is not continuous at $(0,0)$, look at the values of $f(x,y)$ along two different approaches to $(0,0)$, as shown in Figure 24.
- Along the line $y = x$, the limit is

$$\lim_{(x,x)\to(0,0)} f(x,y) = \lim_{(x,x)\to(0,0)} \frac{-3x^2}{2x^2} = -\frac{3}{2}$$

whereas along $y = -x$ you have

$$\lim_{(x,-x)\to(0,0)} f(x,y) = \lim_{(x,-x)\to(0,0)} \frac{3x^2}{2x^2} = \frac{3}{2}.$$

- So, the limit of $f(x,y)$ as $(x,y)\to(0,0)$ does not exist, and you can conclude that f is not continuous at $(0,0)$. Therefore, by Theorem 13.5, you know that f is not differentiable at $(0,0)$.

- On the other hand, by the definition of the partial derivatives f_x and f_y you have

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

and

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0.$$

- So, the partial derivatives at $(0,0)$ exist. ∎

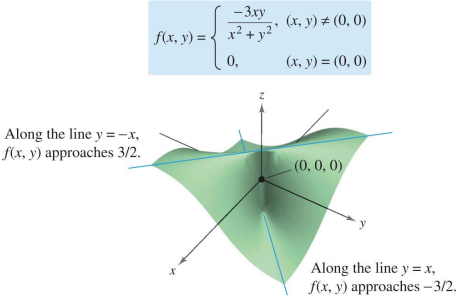

Figure 24: A function not differentiable but partial differential derivatives exist.

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Chain Rules for functions of several variables

Theorem 13.6 (Chain Rule: one independent variable)

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}. \qquad \textit{See Figure 25}$$

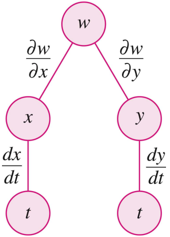

Figure 25: Chain Rule: one independent variable w is a function of x and y, which are each functions of t. It represents the derivative of w with respect to t.

Example 1 (Using the Chain Rule with one independent variable)

Let $w = x^2y - y^2$, where $x = \sin t$ and $y = e^t$. Find $\mathrm{d}w/\mathrm{d}t$ when $t = 0$.

- By the Chain Rule for one independent variable, you have

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = 2xy(\cos t) + (x^2 - 2y)e^t$$
$$= 2(\sin t)(e^t)(\cos t) + (\sin^2 t - 2e^t)e^t = 2e^t \sin t \cos t + e^t \sin^2 t -$$

- When $t = 0$, it follows that

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -2.$$

■

- The Chain Rule in Theorem 13.6 can be extended to any number of variables. For example, if each x_i is a differentiable function of a single variable t, then for

$$w = f(x_1, x_2, \ldots, x_n)$$

you have

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \frac{\partial w}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}t} + \cdots + \frac{\partial w}{\partial x_n}\frac{\mathrm{d}x_n}{\mathrm{d}t}.$$

Example 3 (Finding partial derivatives by substitution)

Find $\partial w/\partial s$ and $\partial w/\partial t$ for $w = 2xy$, where $x = s^2 + t^2$ and $y = s/t$.

- Begin by substituting $x = s^2 + t^2$ and $y = s/t$ into the equation $w = 2xy$ to obtain

$$w = 2xy = 2\left(s^2 + t^2\right)\frac{s}{t} = 2\left(\frac{s^3}{t} + st\right).$$

- Then, to find $\partial w/\partial s$, hold t constant and differentiate with respect to s.

$$\frac{\partial w}{\partial s} = 2\left(\frac{3s^2}{t} + t\right) = \frac{6s^2 + 2t^2}{t}$$

Similarly, to find $\partial w/\partial t$, hold s constant and differentiate with respect to t to obtain

$$\frac{\partial w}{\partial t} = 2\left(-\frac{s^3}{t^2} + s\right) = 2\left(\frac{-s^3 + st^2}{t^2}\right) = \frac{2st^2 - 2s^3}{t^2}. \qquad \blacksquare$$

Theorem 13.7 (Chain Rule: two independent variables)

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(s, t)$ and $y = h(s, t)$ such that the first partial $\frac{\partial x}{\partial s}$, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial s}$, and $\frac{\partial y}{\partial t}$ all exist, then $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

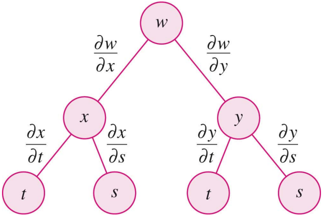

Figure 26: Chain Rule: two independent variables.

Example 4 (The Chain Rule with two independent variables)

Use the Chain Rule to find $\partial w/\partial s$ and $\partial w/\partial t$ for

$$w = 2xy$$

where $x = s^2 + t^2$ and $y = s/t$.

- Using Theorem 13.7, you can hold t constant and differentiate with respect to s to obtain

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = 2y(2s) + 2x\left(\frac{1}{t}\right) \\ &= 2\left(\frac{s}{t}\right)(2s) + 2(s^2+t^2)\left(\frac{1}{t}\right) = \frac{4s^2}{t} + \frac{2s^2+2t^2}{t} = \frac{6s^2+2t^2}{t}.\end{aligned}$$

- Similarly, holding s constant gives

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = 2y(2t) + 2x\left(\frac{-s}{t^2}\right) \\ &= 2\left(\frac{s}{t}\right)(2t) + 2(s^2+t^2)\left(\frac{-s}{t^2}\right) \\ &= 4s - \frac{2s^3+2st^2}{t^2} = \frac{4st^2-2s^3-2st^2}{t^2} = \frac{2st^2-2s^3}{t^2}. \qquad \blacksquare\end{aligned}$$

- The Chain Rule in Theorem 13.7 can also be extended to any number of variables.

- For example, if w is a differentiable function of the n variables x_1, x_2, ..., x_n, where each x_i is a differentiable function of the m variables t_1, t_2, ..., t_m, then for $w = f(x_1, x_2, \ldots, x_n)$ you obtain the following.

$$\begin{aligned}
\frac{\partial w}{\partial t_1} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_1} \\
\frac{\partial w}{\partial t_2} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_2} \\
&\vdots \\
\frac{\partial w}{\partial t_m} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m}
\end{aligned}$$

Example 5 (The Chain Rule for a function of three variables)

Find $\partial w/\partial s$ and $\partial w/\partial t$ when $s = 1$ and $t = 2\pi$ for the function given by $w = xy + yz + xz$ where $x = s\cos t$, $y = s\sin t$, and $z = t$.

- By extending the result of Theorem 13.7, you have

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\ &= (y+z)(\cos t) + (x+z)(\sin t) + (y+x)(0) \\ &= (y+z)(\cos t) + (x+z)(\sin t).\end{aligned}$$

- When $s = 1$ and $t = 2\pi$, you have $x = 1$, $y = 0$, and $z = 2\pi$.
- So, $\partial w/\partial s = (0 + 2\pi)(1) + (1 + 2\pi)(0) = 2\pi$.
- Furthermore,

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \\ &= (y+z)(-s\sin t) + (x+z)(s\cos t) + (y+x)(1)\end{aligned}$$

and for $s = 1$ and $t = 2\pi$, it follows that

$$\frac{\partial w}{\partial t} = (0 + 2\pi)(0) + (1 + 2\pi)(1) + (0 + 1)(1) = 2 + 2\pi. \qquad \blacksquare$$

Implicit partial differentiation

- An application of the Chain Rule to determine the derivative of a function defined implicitly.
- Suppose that x and y are related by the equation $F(x, y) = 0$, where it is assumed that $y = f(x)$ is a differentiable function of x. To find $\mathrm{d}y/\mathrm{d}x$ use Chain Rule. You consider the function given by

$$w = F(x, y) = F(x, f(x))$$

  you can apply Theorem 13.6 to obtain

$$\frac{\mathrm{d}w}{\mathrm{d}x} = F_x(x, y)\,\frac{\mathrm{d}x}{\mathrm{d}x} + F_y(x, y)\,\frac{\mathrm{d}y}{\mathrm{d}x}.$$

- Because $w = F(x, y) = 0$ for all x in the domain of f, you know that $\mathrm{d}w/\mathrm{d}x = 0$ and you have

$$F_x(x, y)\,\frac{\mathrm{d}x}{\mathrm{d}x} + F_y(x, y)\,\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

- Now, if $F_y(x,y) \neq 0$, you can use the fact that $\mathrm{d}x/\mathrm{d}x = 1$ to conclude that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x(x,y)}{F_y(x,y)}.$$
- A similar procedure can be used to find the partial derivatives of functions of several variables that are defined implicitly.

Theorem 13.8 (Chain Rule: implicit differentiation)

If the equation $F(x,y)=0$ defines y implicitly as a differentiable function of x, then
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x(x,y)}{F_y(x,y)}, \quad F_y(x,y) \neq 0.$$
If the equation $F(x,y,z)=0$ defines z implicitly as a differentiable function of x and y, then
$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} \quad \textit{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}, \quad F_z(x,y,z) \neq 0.$$

Example 6 (Finding a derivative implicitly)

Find $\mathrm{d}y/\mathrm{d}x$, given $y^3 + y^2 - 5y - x^2 + 4 = 0$.

- Begin by defining a function F as

$$F(x, y) = y^3 + y^2 - 5y - x^2 + 4 = 0.$$

- Then, using Theorem 13.8, you have

$$F_x(x, y) = -2x \quad \text{and} \quad F_y(x, y) = 3y^2 + 2y - 5$$

and it follows that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x(x, y)}{F_y(x, y)} = \frac{-(-2x)}{3y^2 + 2y - 5} = \frac{2x}{3y^2 + 2y - 5}.$$

■

Example 7 (Finding partial derivatives implicitly)

Find $\partial z/\partial x$ and $\partial z/\partial y$, given $3x^2z - x^2y^2 + 2z^3 + 3yz - 5 = 0$.

- To apply Theorem 13.8, let

$$F(x, y, z) = 3x^2z - x^2y^2 + 2z^3 + 3yz - 5.$$

- Then

$$\begin{aligned} F_x(x, y, z) &= 6xz - 2xy^2 \\ F_y(x, y, z) &= -2x^2y + 3z \\ F_z(x, y, z) &= 3x^2 + 6z^2 + 3y \end{aligned}$$

and you obtain

$$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{F_x(x, y, z)}{F_z(x, y, z)} = \frac{2xy^2 - 6xz}{3x^2 + 6z^2 + 3y} \\ \frac{\partial z}{\partial y} &= -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{2x^2y - 3z}{3x^2 + 6z^2 + 3y}. \end{aligned}$$

■

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Directional derivative

- You are standing on the hillside pictured in Figure 27 and want to determine the hill's incline toward the z-axis.
- If the hill were represented by $z = f(x, y)$, you already know how to determine the slopes in two different directions—the slope in the y-direction would be given by the partial derivative $f_y(x, y)$, and the slope in the x-direction would be given by the partial derivative $f_x(x, y)$.

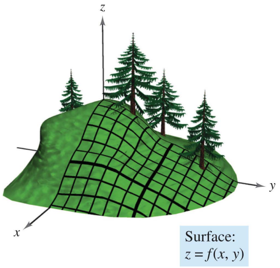
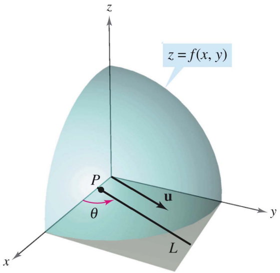


Figure 27: Hill's incline toward the z-axis: Surface $z = f(x, y)$.

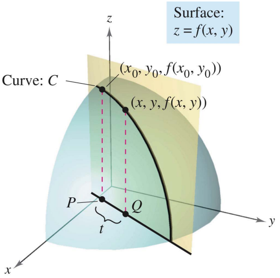
- In this section, you will see that these two partial derivatives can be used to find the slope in any direction. To determine the slope at a point on a surface, you will define a new type of derivative called a directional derivative.
- Begin by letting $z = f(x, y)$ be a surface and $P(x_0, y_0)$ be a point in the domain of f, as shown in Figure 28(a). The "direction" of the directional derivative is given by a unit vector

$$\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

  where θ is the angle the vector makes with the positive x-axis.
- To find the desired slope, reduce the problem to two dimensions by intersecting the surface with a vertical plane passing through the point P and parallel to $\mathbf{u}$, as shown in Figure 28(b).
- This vertical plane intersects the surface to form a curve C.



(a) Line L with direction of $\mathbf{u}$ in xy-plane and surface $z = f(x, y)$.



(b) Curve C on surface $z = f(x, y)$ with projection line L on xy-plane.

Figure 28: Line and curve on surface $z = f(x, y)$.

- The slope of the surface at $(x_0, y_0, f(x_0, y_0))$ in the direction of $\mathbf{u}$ is defined as the slope of the curve C at that point.
- You can write the slope of the curve C as a limit that looks much like those used in single-variable calculus.

- The vertical plane used to form C intersects the xy-plane in a line L, represented by the parametric equations

$$x = x_0 + t\cos\theta \quad \text{and} \quad y = y_0 + t\sin\theta$$

so that for any value of t, the point $Q(x, y)$ lies on the line L.

- For each of the points P and Q, there is a corresponding point on the surface.

$$\begin{array}{ll} (x_0, y_0, f(x_0, y_0)) & \text{Point above } P \\ (x, y, f(x, y)) & \text{Point above } Q \end{array}$$

- Moreover, because the distance between P and Q is

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{(t\cos\theta)^2 + (t\sin\theta)^2} = |t|$$

you can write the slope of the secant line through $(x_0, y_0, f(x_0, y_0))$ and $(x, y, f(x, y))$ as

$$\frac{f(x, y) - f(x_0, y_0)}{t} = \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}.$$

- Finally, by letting t approach 0, you arrive at the following definition.

Definition 13.8 (Directional derivative)

Let f be a function of two variables x and y and let $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ be a unit vector. Then the directional derivative of f in the direction of $\mathbf{u}$, denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x,y) = \lim_{t\to 0} \frac{f(x + t\cos\theta, y + t\sin\theta) - f(x,y)}{t}$$

provided this limit exists.

Theorem 13.9 (Directional derivative)

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta.$$

- For a fixed point (x_0, y_0), let $x = x_0 + t\cos\theta$ and let $y = y_0 + t\sin\theta$. Then, let $g(t) = f(x, y)$. Because f is differentiable, you can apply the Chain Rule given in Theorem 13.6 to obtain

$$g'(t) = f_x(x, y)x'(t) + f_y(x, y)y'(t) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

- If $t = 0$, then $x = x_0$ and $y = y_0$, so

$$g'(0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta.$$

- By the definition of $g'(t)$, it is also true that

$$g'(0) = \lim_{t\to 0}\frac{g(t) - g(0)}{t} = \lim_{t\to 0}\frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}.$$

- Consequently, $D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta$. $\square$

- There are infinitely many directional derivatives of a surface at a given point—one for each direction specified by $\mathbf{u}$, as shown in Figure 29.
- Two of these are the partial derivatives f_x and f_y.

1. Direction of positive x-axis ($\theta = 0$): $\mathbf{u} = \cos 0\,\mathbf{i} + \sin 0\,\mathbf{j} = \mathbf{i}$

$$D_{\mathbf{i}} f(x, y) = f_x(x, y)\cos 0 + f_y(x, y)\sin 0 = f_x(x, y).$$

2. Direction of positive y-axis ($\theta = \pi/2$): $\mathbf{u} = \cos\frac{\pi}{2}\,\mathbf{i} + \sin\frac{\pi}{2}\,\mathbf{j} = \mathbf{j}$

$$D_{\mathbf{j}} f(x, y) = f_x(x, y)\cos\frac{\pi}{2} + f_y(x, y)\sin\frac{\pi}{2} = f_y(x, y).$$

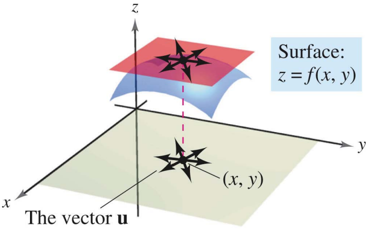


Figure 29: Infinitely many directional derivatives of a surface at a given point.

Example 1 (Finding a directional derivative)

Find the directional derivative of

$$f(x, y) = 4 - x^2 - \frac{1}{4} y^2 \qquad \text{Surface}$$

at $(1, 2)$ in the direction of

$$\mathbf{u} = \left(\cos \frac{\pi}{3}\right) \mathbf{i} + \left(\sin \frac{\pi}{3}\right) \mathbf{j}. \qquad \text{Direction}$$

- Because f_x and f_y are continuous, f is differentiable, and you can apply Theorem 13.9.

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos\theta + f_y(x, y) \sin\theta = (-2x) \cos\theta + \left(-\frac{y}{2}\right) \sin\theta$$

- Evaluating at $\theta = \pi/3$, $x = 1$, and $y = 2$ produces

$$D_{\mathbf{u}}f(1,2) = (-2)\left(\frac{1}{2}\right) + (-1)\left(\frac{\sqrt{3}}{2}\right) = -1 - \frac{\sqrt{3}}{2} \approx -1.866.$$

See Figure 30 ■

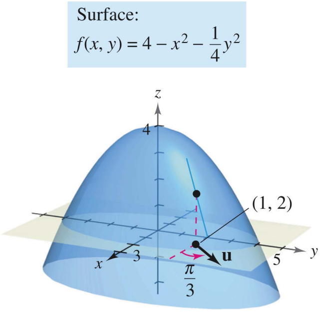


Figure 30: Directional derivative of surface: $f(x, y) = 4 - x^2 - \frac{1}{4}y^2$ at $(1, 2)$ with $\theta = \pi/3$.

Example 2 (Finding a directional derivative)

Find the directional derivative of

$$f(x, y) = x^2 \sin 2y \qquad \text{Surface}$$

at $(1, \pi/2)$ in the direction of

$$\mathbf{v} = 3\,\mathbf{i} - 4\,\mathbf{j}. \qquad \text{Direction}$$

- Because f_x and f_y are continuous, f is differentiable, and you can apply Theorem 13.9.
- Begin by finding a unit vector in the direction of $\mathbf{v}$.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\,\mathbf{i} - \frac{4}{5}\,\mathbf{j} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

- Using this unit vector, you have

$$D_{\mathbf{u}}f(x,y) = (2x\sin 2y)(\cos\theta) + (2x^2\cos 2y)(\sin\theta)$$

$$D_{\mathbf{u}}f\left(1,\frac{\pi}{2}\right) = (2\sin\pi)\left(\frac{3}{5}\right) + (2\cos\pi)\left(-\frac{4}{5}\right)$$

$$= (0)\left(\frac{3}{5}\right) + (-2)\left(-\frac{4}{5}\right) = \frac{8}{5}. \qquad \text{See Figure 31} \qquad \blacksquare$$

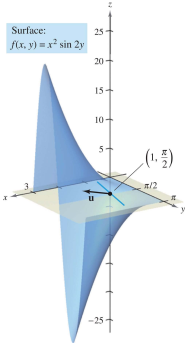


Figure 31: Finding a directional derivative.

The gradient of a function of two variables

- The gradient of a function of two variables is a vector-valued function of two variables.

Definition 13.9 (Gradient of a function of two variables)

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the gradient of f, denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}.$$

∇f is read as "del f". Another notation for the gradient is **grad** $f(x, y)$. In Figure 32, note that for each (x, y), the gradient $\nabla f(x, y)$ is a vector in the plane (not a vector in space).

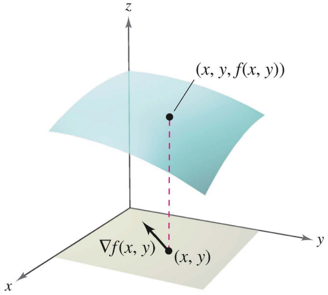


Figure 32: The gradient of f is a vector in the xy-plane.

Example 3 (Finding the gradient of a function)

Find the gradient of $f(x, y) = y \ln x + xy^2$ at the point $(1, 2)$.

- Using

$$f_x(x, y) = \frac{y}{x} + y^2 \quad \text{and} \quad f_y(x, y) = \ln x + 2xy$$

you have

$$\nabla f(x, y) = \left(\frac{y}{x} + y^2\right) \mathbf{i} + (\ln x + 2xy)\, \mathbf{j}.$$

- At the point $(1, 2)$, the gradient is

$$\nabla f(1,2) = \left(\frac{2}{1} + 2^2\right)\mathbf{i} + [\ln 1 + 2(1)(2)]\,\mathbf{j} = 6\,\mathbf{i} + 4\,\mathbf{j}. \qquad \blacksquare$$

Theorem 13.10 (Alternative form of the directional derivative)

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u}.$$

Because the gradient of f is a vector, you can write the directional derivative of f in the direction of $\mathbf{u}$ as

$$D_{\mathbf{u}}f(x,y) = [f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j}] \cdot [\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}].$$

$\square$

Example 4 (Using $\nabla f(x, y)$ to find a directional derivative)

Find the directional derivative of

$$f(x, y) = 3x^2 - 2y^2$$

at $\left(-\frac{3}{4}, 0\right)$ in the direction from $P\left(-\frac{3}{4}, 0\right)$ to $Q(0, 1)$.

- Because the partials of f are continuous, f is differentiable and you can apply Theorem 13.10.
- A vector in the specified direction is

$$\overrightarrow{PQ} = \mathbf{v} = \left(0 + \frac{3}{4}\right)\mathbf{i} + (1 - 0)\mathbf{j} = \frac{3}{4}\mathbf{i} + \mathbf{j}$$

and a unit vector in this direction is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}.$$

- Because $\nabla f(x,y) = f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j} = 6x\,\mathbf{i} - 4y\,\mathbf{j}$, the gradient at $\left(-\frac{3}{4}, 0\right)$ is

$$\nabla f\left(-\frac{3}{4}, 0\right) = -\frac{9}{2}\,\mathbf{i} + 0\,\mathbf{j}.$$

- Consequently, at $\left(-\frac{3}{4}, 0\right)$ the directional derivative is

$$D_{\mathbf{u}} f\left(-\frac{3}{4}, 0\right) = \nabla f\left(-\frac{3}{4}, 0\right) \cdot \mathbf{u} = -\frac{27}{10}. \qquad \blacksquare$$

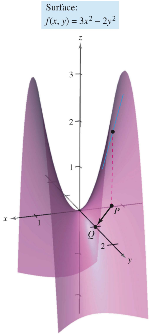


Figure 33: Directional derivative of surface $z = f(x,y) = 3x^2 - 2y^2$.

In many applications, you may want to know in which direction to move so that $f(x, y)$ increases most rapidly. This direction is called the direction of steepest ascent, and it is given by the gradient.

Theorem 13.11 (Properties of the gradient)

Let f be differentiable at the point (x, y).

1. *If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y) = 0$ for all $\mathbf{u}$.*
2. *The direction of maximum increase of f is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}} f(x, y)$ is $\|\nabla f(x, y)\|$.*
3. *The direction of minimum increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}} f(x, y)$ is $-\|\nabla f(x, y)\|$.*

- If $\nabla f(x, y) = \mathbf{0}$, then for any direction (any $\mathbf{u}$), you have

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = (0\,\mathbf{i} + 0\,\mathbf{j}) \cdot (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) = 0.$$

- If $\nabla f(x, y) \neq \mathbf{0}$, then let ϕ be the angle between $\nabla f(x, y)$ and a unit vector $\mathbf{u}$. Using the dot product, you can apply Theorem 11.5 to conclude that

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = \|\nabla f(x, y)\| \; \|\mathbf{u}\| \cos\phi = \|\nabla f(x, y)\| \cos\phi$$

  and it follows that the maximum value of $D_{\mathbf{u}} f(x, y)$ will occur when $\cos\phi = 1$.

- So, $\phi = 0$, and the maximum value of the directional derivative occurs when $\mathbf{u}$ has the same direction as $\nabla f(x, y)$.

- Moreover, this largest value of $D_{\mathbf{u}} f(x, y)$ is precisely
$$\|\nabla f(x, y)\| \cos \phi = \|\nabla f(x, y)\| .$$
- Similarly, the minimum value of $D_{\mathbf{u}} f(x, y)$ can be obtained by letting $\phi = \pi$ so that $\mathbf{u}$ points in the direction opposite that of $\nabla f(x, y)$, as shown in Figure 34. $\square$

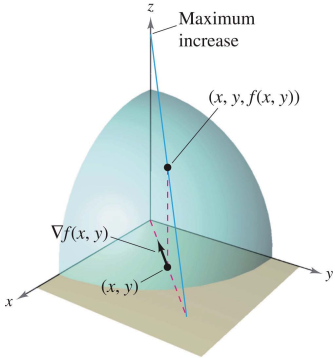


Figure 34: The gradient of f is a vector in the xy-plane that points in the direction of maximum increase on the surface given by $z = f(x, y)$.

Example 5 (Finding the direction of maximum increase)

The temperature in degrees Celsius on the surface of a metal plate is

$$T(x, y) = 20 - 4x^2 - y^2$$

where x and y are measured in centimeters. In what direction from $(2, -3)$ does the temperature increase most rapidly? What is this rate of increase?

- The gradient is

$$\nabla T(x, y) = T_x(x, y)\,\mathbf{i} + T_y(x, y)\,\mathbf{j} = -8x\,\mathbf{i} - 2y\,\mathbf{j}.$$

- It follows that the direction of maximum increase is given by

$$\nabla T(2, -3) = -16\,\mathbf{i} + 6\,\mathbf{j}$$

as shown in Figure 35, and the rate of increase is

$$\|\nabla T(2, -3)\| = \sqrt{256 + 36} = \sqrt{292} \approx 17.09^\circ \text{ (per centimeter)} \quad \blacksquare$$

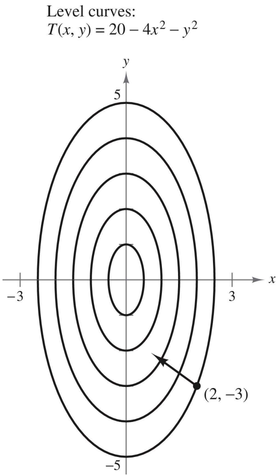


Figure 35: The direction of the most rapid increase in temperature in $(2, -3)$ is given by $-16\,\mathbf{i} + 6\,\mathbf{j}$.

Example 6 (Finding the path of a heat-seeking particle)

A heat-seeking particle is located at the point $(2, -3)$ on a metal plate whose temperature at (x, y) is

$$T(x, y) = 20 - 4x^2 - y^2.$$

Find the path of the particle as it continuously moves in the direction of maximum temperature increase.

- Let the path be represented by the position function

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}.$$

- A tangent vector at each point $(x(t), y(t))$ is given by

$$\mathbf{r}'(t) = \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathbf{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathbf{j}.$$

- Because the particle seeks maximum temperature increase, the directions of $\mathbf{r}'(t)$ and $\nabla T(x, y) = -8x\,\mathbf{i} - 2y\,\mathbf{j}$ are the same at each point on the path.

- So,
$$-8x = k\,\frac{\mathrm{d}x}{\mathrm{d}t} \quad \text{and} \quad -2y = k\,\frac{\mathrm{d}y}{\mathrm{d}t}$$
where k depends on t.
- By solving each equation for $\mathrm{d}t/k$ and equating the results, you obtain
$$\frac{\mathrm{d}x}{-8x} = \frac{\mathrm{d}y}{-2y}.$$
- The solution of this differential equation is $x = Cy^4$.
- Because the particle starts at the point $(2, -3)$, you can determine that $C = 2/81$. So, the path of the heat-seeking particle is
$$x = \frac{2}{81}\,y^4.$$

■

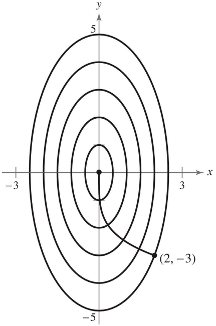


Figure 36: Path followed by a heat-seeking particle.

Theorem 13.12 (Gradient is normal to level curves)

If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).

- Consider that the $z = f(x, y)$ is constant along a given level curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle$.
- So, at any point (x, y) on the curve, the rate of change of $f(x, y)$ in the direction of a unit tangent vector $\mathbf{u} = \mathbf{T}(t) = \mathbf{r}'(t)/\left\|\mathbf{r}'(t)\right\|$ is 0, and you can write

$$z'(t) = \nabla f(x, y) \cdot \mathbf{u} = D_{\mathbf{u}} f(x, y) = 0.$$

$\square$

Example 7 (Finding a normal vector to a level curve)

Sketch the level curve corresponding to $c = 0$ for the function given by $f(x, y) = y - \sin x$ and find a normal vector at several points on the curve.

- The level curve for $c = 0$ is given by

$$0 = y - \sin x \qquad\qquad y = \sin x$$

as shown in Figure 37(a).

- Because the gradient vector of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j} = -\cos x\,\mathbf{i} + \mathbf{j}$$

you can use Theorem 13.12 to conclude that $\nabla f(x, y)$ is normal to the level curve at the point (x, y).

- Some gradient vectors are

$$\nabla f(-\pi, 0) = \mathbf{i} + \mathbf{j} \qquad \nabla f\left(-\frac{2\pi}{3}, -\frac{\sqrt{3}}{2}\right) = \frac{1}{2}\,\mathbf{i} + \mathbf{j}$$

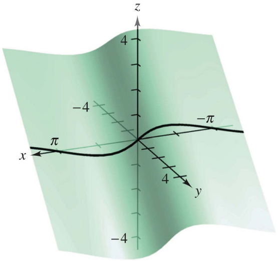
$$\nabla f\left(-\frac{\pi}{2}, -1\right) = \mathbf{j} \qquad \nabla f\left(-\frac{\pi}{3}, -\frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\,\mathbf{i} + \mathbf{j}$$

$$\nabla f(0, 0) = -\mathbf{i} + \mathbf{j} \qquad \nabla f\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\,\mathbf{i} + \mathbf{j}$$

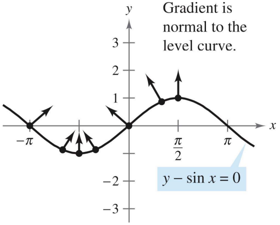
$$\nabla f\left(\frac{\pi}{2}, 1\right) = \mathbf{j} \qquad \nabla f\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right) = \frac{1}{2}\mathbf{i} + \mathbf{j}$$

$$\nabla f(\pi, 0) = \mathbf{i} + \mathbf{j}.$$

- These are shown in Figure 37(b). ■



(a) The surface is given by $f(x, y) = y - \sin x$.



(b) The level curve is given by $f(x, y) = 0$.

Figure 37: Finding a normal vector to a level curve.

Definition 13.10 (Directional derivative and gradient for three variables)

Let f be a function of x, y, and z, with continuous first partial derivatives. The directional derivative of f in the direction of a unit vector $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ is given by

$$D_{\mathbf{u}}f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z).$$

The gradient of f is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}.$$

Definition 13.10

Properties of the gradient are as follows.

1. $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y, z) = 0$ for all $\mathbf{u}$.
3. The direction of maximum increase of f is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}} f(x, y, z)$ is

$$\|\nabla f(x, y, z)\| . \qquad \text{Maximum value of } D_{\mathbf{u}} f(x, y, z)$$

4. The direction of minimum increase of f is given by $-\nabla f(x, y, z)$. The minimum value of $D_{\mathbf{u}} f(x, y, z)$ is

$$-\|\nabla f(x, y, z)\| . \qquad \text{Minimum value of } D_{\mathbf{u}} f(x, y, z)$$

Example 8 (Finding the gradient for a function of three variables)

Find $\nabla f(x, y, z)$ for the function given by

$$f(x, y, z) = x^2 + y^2 - 4z$$

and find the direction of maximum increase of f at the point $(2, -1, 1)$.

- The gradient vector is given by

$$\begin{aligned} \nabla f(x, y, z) &= f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k} \\ &= 2x\,\mathbf{i} + 2y\,\mathbf{j} - 4\,\mathbf{k}. \end{aligned}$$

- So, it follows that the direction of maximum increase at $(2, -1, 1)$ is

$$\nabla f(2, -1, 1) = 4\,\mathbf{i} - 2\,\mathbf{j} - 4\,\mathbf{k} \qquad \text{see Figure 38.}$$

■

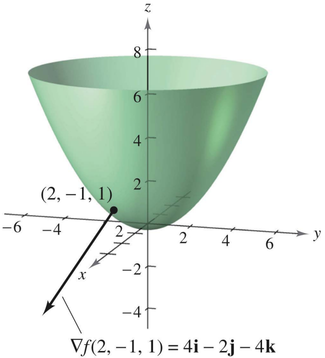


Figure 38: Level surface and gradient vector at $(2, -1, 1)$ for $f(x, y, z) = x^2 + y^2 - 4z$.

Table of Contents

Tangent plane and normal line to a surface

- You can represent the surfaces in space primarily by equations of the form
$$z = f(x, y). \qquad \text{Equation of a surface } S$$
- In the development to follow, however, it is convenient to use the more general representation $F(x, y, z) = 0$.
- For a surface S given by $z = f(x, y)$, you can convert to the general form by defining F as $F(x, y, z) = f(x, y) - z$.
- Because $f(x, y) - z = 0$, you can consider S to be the level surface of F given by
$$F(x, y, z) = 0. \qquad \text{Alternative equation of surface } S$$

Example 1 (Writing an equation of a surface)

For the function given by

$$F(x, y, z) = x^2 + y^2 + z^2 - 4$$

describe the level surface given by $F(x, y, z) = 0$.

- The level surface given by $F(x, y, z) = 0$ can be written as

$$x^2 + y^2 + z^2 = 4$$

  which is a sphere of radius 2 whose center is at the origin. ■

- Normal lines are important in analyzing surfaces and solids. For example, consider the collision of two billiard balls.
- When a stationary ball is struck at a point P on its surface, it moves along the line of impact determined by P and the center of the ball.
- The impact can occur in two ways.

- If the cue ball is moving along the line of impact, it stops dead and imparts all of its momentum to the stationary ball.

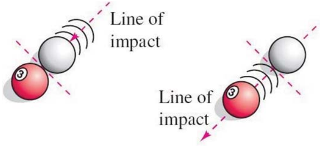


Figure 39: Line of impact

- If the cue ball is not moving along the line of impact, it is deflected to one side or the other and retains part of its momentum.
- That part of the momentum that is transferred to the stationary ball occurs along the line of impact, regardless of the direction of the cue ball. This line of impact is called the normal line to the surface of the ball at the point P.

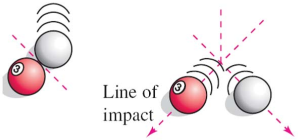


Figure 40: Line of impact.

- In the process of finding a normal line to a surface, you are also able to solve the problem of finding a tangent plane to the surface.
- Let S be a surface given by

$$F(x, y, z) = 0$$

  and let $P(x_0, y_0, z_0)$ be a point on S.
- Let C be a curve on S through P that is defined by the vector-valued function

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}.$$

- Then, for all t,

$$F(x(t), y(t), z(t)) = 0.$$

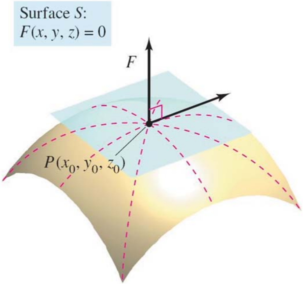
- If F is differentiable and $x'(t)$, $y'(t)$, and $z'(t)$ all exist, it follows from the Chain Rule that

$$0 = F'(t) = F_x(x, y, z)x'(t) + F_y(x, y, z)y'(t) + F_z(x, y, z)z'(t).$$

- At (x_0, y_0, z_0), the equivalent vector form is

$$0 = \underbrace{\nabla F(x_0, y_0, z_0)}_{\text{Gradient}} \cdot \underbrace{\mathbf{r}'(t_0)}_{\text{Tangent vector}}.$$

- This result means that the gradient at P is orthogonal to the tangent vector of every curve on S through P. So, all tangent lines on S lie in a plane that is normal to $\nabla F(x_0, y_0, z_0)$ and contains P, as below.



Definition 13.11 (Tangent plane and normal line)

Let F be differentiable at the point $P(x_0, y_0, z_0)$ on the surface S given by $F(x, y, z) = 0$ such that $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$.

1. The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the tangent plane to S at P.
2. The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called the normal line to S at P.

Theorem 13.13 (Equation of tangent plane)

If F is differentiable at (x_0, y_0, z_0), then an equation of the tangent plane to the surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Example 2 (Finding an equation of a tangent plane)

Find an equation of the tangent plane to the hyperboloid given by

$$z^2 - 2x^2 - 2y^2 = 12$$

at the point $(1, -1, 4)$.

- Begin by writing the equation of the surface as

$$z^2 - 2x^2 - 2y^2 - 12 = 0.$$

- Then, considering

$$F(x, y, z) = z^2 - 2x^2 - 2y^2 - 12$$

you have $F_x(x, y, z) = -4x$, $F_y(x, y, z) = -4y$, and $F_z(x, y, z) = 2z$.

- At the point $(1, -1, 4)$ the partial derivatives are

$$F_x(1, -1, 4) = -4, \quad F_y(1, -1, 4) = 4, \quad \text{and} \quad F_z(1, -1, 4) = 8.$$

- So, an equation of the tangent plane at $(1, -1, 4)$ is

$$-4(x-1) + 4(y+1) + 8(z-4) = 0$$
$$x - y - 2z + 6 = 0.$$

- Figure 42 shows a portion of the hyperboloid and tangent plane. ■

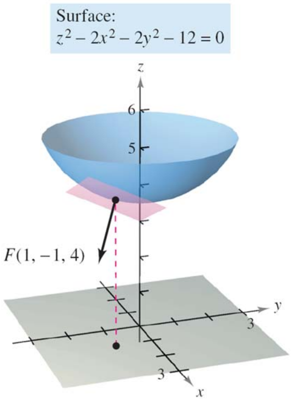


Figure 42: Tangent plane to surface: $z^2 - 2x^2 - 2y^2 - 12 = 0$.

- To find the equation of the tangent plane at a point on a surface given by $z = f(x, y)$, you can define the function F by

$$F(x, y, z) = f(x, y) - z.$$

- Then S is given by the level surface $F(x, y, z) = 0$, and by Theorem 13.13 an equation of the tangent plane to S at the point (x_0, y_0, z_0) is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Example 3 (Finding an equation of the tangent plane)

Find the equation of the tangent plane to the paraboloid

$$z = 1 - \frac{1}{10}(x^2 + 4y^2)$$

at the point $\left(1, 1, \frac{1}{2}\right)$.

- From $z = f(x, y) = 1 - \frac{1}{10}(x^2 + 4y^2)$, you obtain

$$f_x(x, y) = -\frac{x}{5} \quad \Longrightarrow \quad f_x(1, 1) = -\frac{1}{5} \quad \text{and}$$
$$f_y(x, y) = -\frac{4y}{5} \quad \Longrightarrow \quad f_y(1, 1) = -\frac{4}{5}.$$

- So, an equation of the tangent plane at $\left(1, 1, \frac{1}{2}\right)$ is

$$\begin{aligned} f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) - \left(z - \frac{1}{2}\right) &= 0 \\ -\frac{1}{5}(x - 1) - \frac{4}{5}(y - 1) - \left(z - \frac{1}{2}\right) &= 0 \\ -\frac{1}{5}x - \frac{4}{5}y - z + \frac{3}{2} &= 0. \end{aligned}$$

- This tangent plane is shown in Figure 43. ■

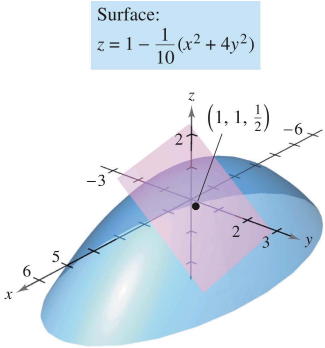


Figure 43: Finding an equation of the tangent plane.

Example 4 (Finding an equation of a normal line to a surface)

Find a set of symmetric equations for the normal line to the surface given by $xyz = 12$ at the point $(2, -2, -3)$.

- Begin by letting

$$F(x, y, z) = xyz - 12.$$

- Then, the gradient is given by

$$\begin{aligned} \nabla F(x, y, z) &= F_x(x, y, z)\,\mathbf{i} + F_y(x, y, z)\,\mathbf{j} + F_z(x, y, z)\,\mathbf{k} \\ &= yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k} \end{aligned}$$

and at the point $(2, -2, -3)$ you have

$$\nabla F(2, -2, -3) = (-2)(-3)\,\mathbf{i} + (2)(-3)\,\mathbf{j} + (2)(-2)\,\mathbf{k} = 6\,\mathbf{i} - 6\,\mathbf{j} - 4\,\mathbf{k}.$$

- The normal line at $(2, -2, -3)$ has direction numbers 6, -6, and -4, and the corresponding set of symmetric equations is

$$\frac{x-2}{6} = \frac{y+2}{-6} = \frac{z+3}{-4}.$$

- See Figure 44. ■

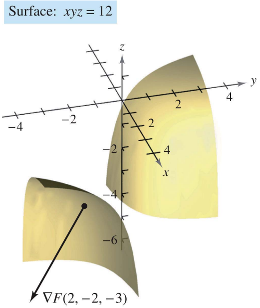


Figure 44: Finding an equation of a normal line to a surface.

Example 5 (Finding the equation of a tangent line to a curve)

Describe the tangent line to the curve of intersection of the surfaces

$$x^2 + 2y^2 + 2z^2 = 20 \qquad \text{Ellipsoid}$$
$$x^2 + y^2 + z = 4 \qquad \text{Paraboloid}$$

at the point $(0, 1, 3)$, as shown in Figure 45.

- Begin by finding the gradients to both surfaces at the point $(0,1,3)$.

Ellipsoid	Paraboloid
$F(x,y,z) = x^2+2y^2+2z^2-20$	$G(x,y,z) = x^2+y^2+z-4$
$\nabla F(x,y,z) = 2x\,\mathbf{i}+4y\,\mathbf{j}+4z\,\mathbf{k}$	$\nabla G(x,y,z) = 2x\,\mathbf{i}+2y\,\mathbf{j}+\mathbf{k}$
$\nabla F(0,1,3) = 4\,\mathbf{j}+12\,\mathbf{k}$	$\nabla G(0,1,3) = 2\,\mathbf{j}+\mathbf{k}$

- The cross product of these two gradients is a vector that is tangent to both surfaces at the point $(0,1,3)$.

$$\nabla F(0,1,3)\times\nabla G(0,1,3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 4 & 12 \\ 0 & 2 & 1 \end{vmatrix} = -20\,\mathbf{i}$$

- So, the tangent line to the curve of intersection of the two surfaces at the point $(0,1,3)$ is a line that is parallel to the x-axis and passes through the point $(0,1,3)$. ■

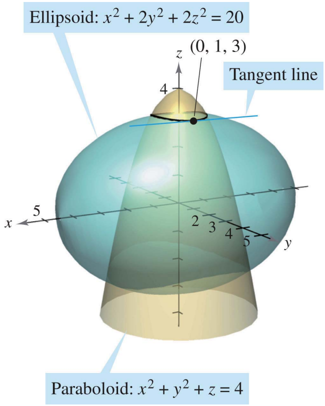


Figure 45: Finding the equation of a tangent line to a curve.

A comparison of the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$

- This section concludes with a comparison of the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$.
- You know that the gradient of a function f of two variables is normal to the level curves of f.
- Specifically, Theorem 13.12 states that if f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).
- Having developed normal lines to surfaces, you can now extend this result to a function of three variables.

Theorem 13.14 (Gradient is normal to level surfaces)

If F is differentiable at (x_0, y_0, z_0) and $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\nabla F(x_0, y_0, z_0)$ is normal to the level surface through (x_0, y_0, z_0).

- Consider that the $w = F(x, y, z)$ is constant along a given level surface S through (x_0, y_0, z_0) and $\mathbf{r}(t) = (x(t), y(t), z(t))$ on S through $\mathbf{r}(0) = (x_0, y_0, z_0)$.
- So, at any point (x_0, y_0, z_0) on the surface, the rate of change of $F(x, y, z)$ is 0, and you can write

$$w'(0) = \nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(0) = 0.$$

□

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Absolute extrema and relative extrema

- Consider the continuous function f of two variables, defined on a closed bounded region R.
- The values $f(a, b)$ and $f(c, d)$ such that

$$f(a, b) \leq f(x, y) \leq f(c, d) \qquad (a, b) \text{ and } (c, d) \text{ are in } R$$

  for all (x, y) in R are called the minimum and maximum of f in the region R, as shown in Figure 46.
- A region in the plane is closed if it contains all of its boundary points. The Extreme Value Theorem deals with a region in the plane that is both closed and bounded.
- A region in the plane is called boundedif it is a subregion of a closed disk in the plane.

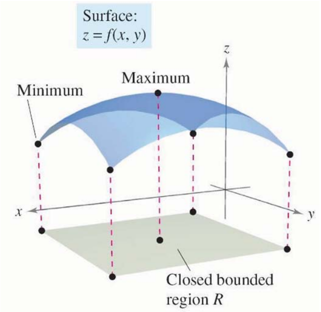


Figure 46: R contains point(s) at which $f(x, y)$ is a minimum and point(s) at which $f(x, y)$ is a maximum.

Theorem 13.15 (Extreme Value Theorem)

Let f be a continuous function of two variables x and y defined on a closed bounded region R in the xy-plane.

1. *There is at least one point in R at which f takes on a minimum value.*
2. *There is at least one point in R at which f takes on a maximum value.*

- A minimum is also called an absolute minimum and a maximum is also called an absolute maximum.
- As in single-variable calculus, there is a distinction made between absolute extrema and relative extrema.

Definition 13.12 (Relative extrema)

Let f be a function defined on a region R containing (x_0, y_0).

1. The function f has a relative minimum at (x_0, y_0) if
$$f(x, y) \geq f(x_0, y_0)$$
for all (x, y) in an open disk containing (x_0, y_0).
2. The function f has a relative maximum at (x_0, y_0) if
$$f(x, y) \leq f(x_0, y_0)$$
for all (x, y) in an open disk containing (x_0, y_0).

- To say that f has a relative maximum at (x_0, y_0) means that the point (x_0, y_0, z_0) is at least as high as all nearby points on the graph of $z = f(x, y)$.
- Similarly, f has a relative minimum at (x_0, y_0) if (x_0, y_0, z_0) is at least as low as all nearby points on the graph. (See Figure 47.)
- To locate relative extrema of f, you can investigate the points at which the gradient of f is $\mathbf{0}$ or the points at which one of the partial derivatives does not exist. Such points are called critical point of f.

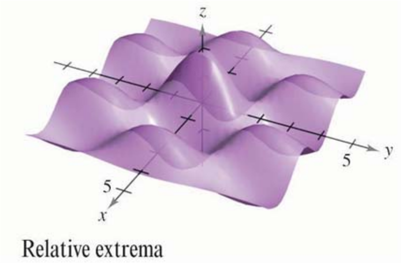


Figure 47: Relative extrema.

Definition 13.13 (Critical point)

Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a critical point of f if one of the following is true.

1. $f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

- If f is differentiable and

$$\nabla f(x_0, y_0) = f_x(x_0 y_0)\,\mathbf{i} + f_y(x_0, y_0)\,\mathbf{j} = 0\,\mathbf{i} + 0\,\mathbf{j}$$

  then every directional derivative at (x_0, y_0) must be 0.
- This implies that the function has a horizontal tangent plane at the point (x_0, y_0), as shown in Figure 48.
- It appears that such a point is a likely location of a relative extremum.
- This is confirmed by Theorem 13.16.

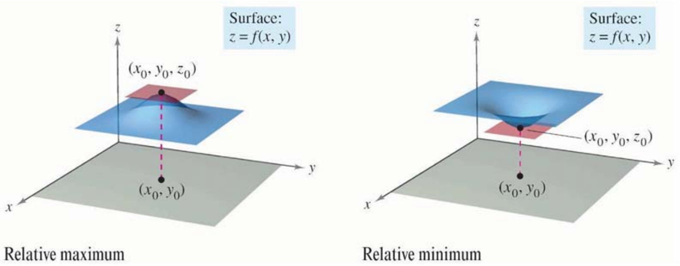


Figure 48: Relative extrema.

Theorem 13.16 (Relative extrema occur only at critical points)

If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.

Example 1 (Finding a relative extremum)

Determine the relative extrema of

$$f(x,y) = 2x^2 + y^2 + 8x - 6y + 20.$$

- Begin by finding the critical points of f. Because

$$f_x(x,y) = 4x + 8 \quad \text{and} \quad f_y(x,y) = 2y - 6$$

  are defined for all x and y, the only critical points are those for which both first partial derivatives are 0.
- To locate these points, set $f_x(x,y)$ and $f_y(x,y)$ equal to 0, and solve the equations

$$4x + 8 = 0 \quad \text{and} \quad 2y - 6 = 0$$

  to obtain the critical point $(-2,3)$.
- By completing the square, you can conclude that for all $(x,y) \neq (-2,3)$

$$f(x,y) = 2(x+2)^2 + (y-3)^2 + 3 > 3.$$

- So, a relative minimum of f occurs at $(-2, 3)$.
- The value of the relative minimum is $f(-2, 3) = 3$, as shown below. ■

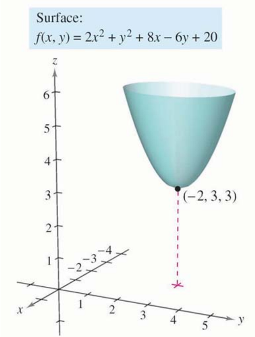


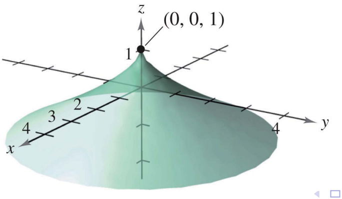
Figure 49: The function $z = f(x, y)$ has a relative minimum at $(-2, 3)$.

Example 2 (Finding a relative extremum)

Determine the relative extrema of $f(x, y) = 1 - (x^2 + y^2)^{1/3}$.

- Because
$$f_x(x,y) = -\frac{2x}{3(x^2+y^2)^{2/3}} \quad \text{and} \quad f_y(x,y) = -\frac{2y}{3(x^2+y^2)^{2/3}}$$
it follows that both partial derivatives exist for all points in the xy-plane except for $(0,0)$. Moreover, because the partial derivatives cannot both be 0 unless both x and y are 0, you can conclude that $(0,0)$ is the only critical point.
- In Figure below, note that $f(0,0)$ is 1. For all other (x,y) it is clear
$$f(x,y) = 1 - (x^2+y^2)^{1/3} < 1.$$
So, f has a relative maximum at $(0,0)$. ■

Surface:
$f(x, y) = 1 - (x^2 + y^2)^{1/3}$



The second partials test

- To find relative extrema you need only examine values of $f(x, y)$ at critical points.
- However, as is true for a function of one variable, the critical points of a function of two variables do not always yield relative maxima or minima. Some critical points yield saddle points, which are neither relative maxima nor relative minima.
- As an example of a critical point that does not yield a relative extremum, consider the surface given by

$$f(x, y) = y^2 - x^2 \qquad \text{Hyperbolic paraboloid}$$

  as shown in Figure 50.
- At the point $(0, 0)$, both partial derivatives are 0.

- The function f does not, however, have a relative extremum at this point because in any open disk centered at $(0, 0)$ the function takes on both negative values (along the x-axis) and positive values (along the y-axis).
- So, the point $(0, 0, 0)$ is a saddle point of the surface.

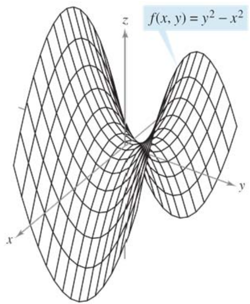


Figure 50: Saddle point at $(0, 0, 0)$: $f_x(0, 0) = f_y(0, 0) = 0$ where $f(x, y) = y^2 - x^2$.

Theorem 13.17 (Second Partials Test)

Let f have continuous second partial derivatives on an open region containing a point (a, b) for which

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

To test for relative extrema of f, consider the quantity $d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$.

1. *If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a relative minimum at (a, b).*
2. *If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a relative maximum at (a, b).*
3. *If $d < 0$, then $(a, b, f(a, b))$ is a saddle point.*
4. *The test is inconclusive if $d = 0$.*

Example 3 (Using the Second Partials Test)

Find the relative extrema of

$$f(x, y) = -x^3 + 4xy - 2y^2 + 1.$$

- Begin by finding the critical points of f. Because

$$f_x(x, y) = -3x^2 + 4y \quad \text{and} \quad f_y(x, y) = 4x - 4y$$

  exist for all x and y, the only critical points are those for which both first partial derivatives are 0.

- To locate these points, set $f_x(x, y)$ and $f_y(x, y)$ equal to 0 to obtain

$$-3x^2 + 4y = 0 \quad \text{and} \quad 4x - 4y = 0.$$

- From the second equation you know that $x = y$, and, by substitution into the first equation, you obtain two solutions: $y = x = 0$ and $y = x = \frac{4}{3}$.

- Because

$$f_{xx}(x,y) = -6x, \quad f_{yy}(x,y) = -4, \quad \text{and} \quad f_{xy}(x,y) = 4$$

it follows that, for the critical point $(0,0)$,

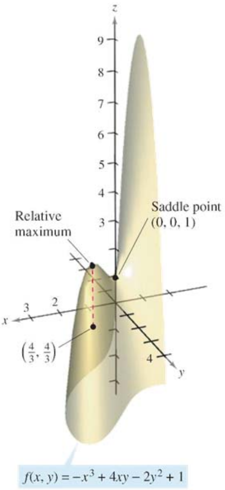
$$d = f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2 = 0 - 16 < 0$$

and, by the Second Partials Test, you can conclude that $(0,0,1)$ is a saddle point of f.

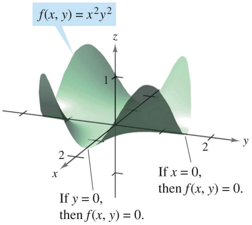
- Furthermore, for the critical point $\left(\frac{4}{3}, \frac{4}{3}\right)$,

$$d = f_{xx}\left(\frac{4}{3}, \frac{4}{3}\right) f_{yy}\left(\frac{4}{3}, \frac{4}{3}\right) - \left[f_{xy}\left(\frac{4}{3}, \frac{4}{3}\right)\right]^2 = -8(-4) - 16 = 16 > 0$$

and because $f_{xx}\left(\frac{4}{3}, \frac{4}{3}\right) = -8 < 0$ you can conclude that f has a relative maximum at $\left(\frac{4}{3}, \frac{4}{3}\right)$ as shown in Figure 51. ■



(a) sing the Second Partials Test.



(b) Failure of the Second Partials Test.

Figure 51: Second Partials Test.

Example 4 (Failure of the Second Partials Test)

Find the relative extrema of $f(x, y) = x^2y^2$.

- Because $f_x(x, y) = 2xy^2$ and $f_y(x, y) = 2x^2y$, you know that both partial derivatives are 0 if $x = 0$ or $y = 0$. That is, every point along the x- or y-axis is a critical point.
- Moreover, because

$$f_{xx}(x, y) = 2y^2, \qquad f_{yy}(x, y) = 2x^2, \quad \text{and} \quad f_{xy}(x, y) = 4xy$$

if either $x = 0$ or $y = 0$, then

$$d = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 = 4x^2y^2 - 16x^2y^2 = -12x^2y^2.$$

- So, the Second Partials Test fails. However, because $f(x, y) = 0$ for every point along the x- or y-axis and $f(x, y) = x^2y^2 > 0$ for all other points, you can conclude that each of these critical points yields an absolute minimum, as shown in Figure 51. ■

The Second Partials Test can fail to find relative extrema in two ways. If either of the first partial derivatives does not exist, you cannot use the test. Also, if

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 = 0$$

the test fails. In such cases, you can try a sketch or some other approach.

Example 5 (Finding absolute extrema)

Find the absolute extrema of the function

$$f(x, y) = \sin xy$$

on the closed region given by $0 \leq x \leq \pi$ and $0 \leq y \leq 1$.

- From the partial derivatives

$$f_x(x, y) = y \cos xy \quad \text{and} \quad f_y(x, y) = x \cos xy$$

you can see that each point lying on the hyperbola given by $xy = \pi/2$ is a critical point. These points each yield the value

$$f(x, y) = \sin \frac{\pi}{2} = 1$$

which you know is the absolute maximum.

- The only other critical point of f lying in the given region is $(0, 0)$. It yields an absolute minimum of 0, because

$$0 \leq xy \leq \pi \quad \text{implies that} \quad 0 \leq \sin xy \leq 1.$$

- To locate other absolute extrema, you should consider the four boundaries of the region formed by taking traces with the vertical planes $x = 0$, $x = \pi$, $y = 0$, and $y = 1$. In doing this, you will find that $\sin xy = 0$ at all points on the x-axis, at all points on the y-axis, and at the point $(\pi, 1)$. Each of these points yields an absolute minimum for the surface. ■

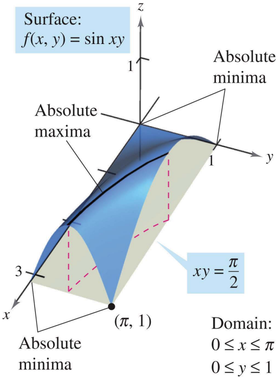


Figure 52: Finding absolute extrema.

- Absolute extrema of a function can occur in two ways.
  - First, some relative extrema also happen to be absolute extrema. For instance, in Example 1, $f(-2, 3)$ is an absolute minimum of the function. (On the other hand, the relative maximum found in Example 3 is not an absolute maximum of the function.)
  - Second, absolute extrema can occur at a boundary point of the domain.

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- Many optimization problems have restrictions, or constraints, on the values that can be used to produce the optimal solution. Such constraints tend to complicate optimization problems because the optimal solution can occur at a boundary point of the domain.
- In this section, you will study an ingenious technique for solving such problems. It is called the Method of Lagrange Multipliers.
- To see how this technique works, suppose you want to find the rectangle of maximum area that can be inscribed in the ellipse given by $\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$.
- Let (x, y) be the vertex of the rectangle in the first quadrant, as shown in Figure 53.

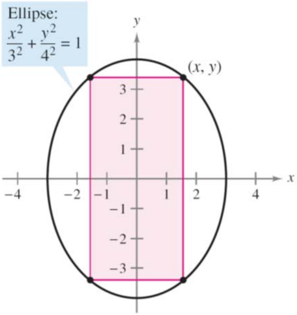

Figure 53: Objective function: $f(x, y) = 4xy$ and constrain: $\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$.

- Because the rectangle has sides of lengths $2x$ and $2y$, its area is given by

$$f(x, y) = 4xy. \qquad \text{Objective function}$$

- You want to find x and y such that $f(x, y)$ is a maximum.

- Your choice of (x, y) is restricted to first-quadrant points that lie on the ellipse

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1. \qquad \text{Constraint}$$

- Now, consider the constraint equation to be a fixed level curve of

$$g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2}.$$

- The level curves of f represent a family of hyperbolas $f(x, y) = 4xy = k$. In this family, the level curves that meet the given constraint correspond to the hyperbolas that intersect the ellipse.
- Moreover, to maximize $f(x, y)$, you want to find the hyperbola that just barely satisfies the constraint. The level curve that does this is the one that is tangent to the ellipse, as shown in Figure 54.

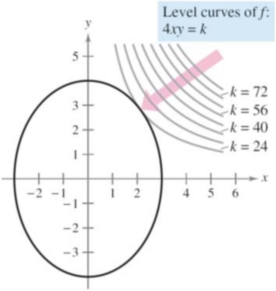

Figure 54: Level curves of f: $4xy = k$; Constraint: $g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$.

- If $\nabla f(x, y) = \lambda \nabla g(x, y)$, then scalar λ is called Lagrange Multiplier.

Theorem 13.18 (Lagrange's Theorem)

Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

- To begin, represent the smooth curve given by $g(x, y) = c$ by the vector-valued function

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}, \quad \mathbf{r}'(t) \neq \mathbf{0}$$

  where x' and y' are continuous on an open interval I.
- Define the function h as $h(t) = f(x(t), y(t))$. Then, because $f(x_0, y_0)$ is an extreme value of f, you know that

$$h(t_0) = f(x(t_0), y(t_0)) = f(x_0, y_0)$$

  is an extreme value of h.

- This implies that $h'(t_0) = 0$, and, by the Chain Rule,

$$h'(t_0) = f_x(x_0, y_0)x'(t_0) + f_y(x_0, y_0)y'(t_0) = \nabla f(x_0, y_0) \cdot \mathbf{r}'(t_0) = 0.$$

  So, $\nabla f(x_0, y_0)$ is orthogonal to $\mathbf{r}'(t_0)$. Moreover, by Theorem 13.12, $\nabla g(x_0, y_0)$ is also orthogonal to $\mathbf{r}'(t_0)$.

- Consequently, the gradients $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are parallel, and there must exist a scalar λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

□

Method of Lagrange Multipliers Let f and g satisfy the hypothesis of Lagrange's Theorem 13.18, and let f have a minimum or maximum subject to the constraint $g(x, y) = c$. To find the minimum or maximum of f, use the following steps.

1. Simultaneously solve the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = c$ by solving the following system of equations.

$$f_x(x, y) = \lambda g_x(x, y) \qquad f_y(x, y) = \lambda g_y(x, y) \qquad g(x, y) = c$$

2. Evaluate f at each solution point obtained in the first step. The largest value yields the maximum of f subject to the constraint $g(x, y) = c$, and the smallest value yields the minimum of f subject to the constraint $g(x, y) = c$.

Alternative: Let $F(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$. Then solve the free-constrained optimization problem for F.

Example 1 (Using a Lagrange Multiplier with one constraint)

Find the maximum value of $f(x,y) = 4xy$ where $x > 0$ and $y > 0$, subject to the constraint $(x^2/3^2) + (y^2/4^2) = 1$.

- To begin, let
$$g(x,y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$
- By equating $\nabla f(x,y) = 4y\,\mathbf{i} + 4x\,\mathbf{j}$ and $\lambda \nabla g(x,y) = (2\lambda x/9)\,\mathbf{i} + (\lambda y/8)\,\mathbf{j}$, you can obtain the following system of equations.

$$\begin{aligned} 4y &= \frac{2}{9}\lambda x && f_x(x,y) = \lambda g_x(x,y) \\ 4x &= \frac{1}{8}\lambda y && f_y(x,y) = \lambda g_y(x,y) \\ \frac{x^2}{3^2} + \frac{y^2}{4^2} &= 1 && \text{Constraint} \end{aligned}$$

- From the first equation, you obtain $\lambda = 18y/x$, and substitution into the second equation produces

$$4x = \frac{1}{8}\left(\frac{18y}{x}\right) y \quad \Longrightarrow \quad x^2 = \frac{9}{16}\, y^2.$$

- Substituting this value for x^2 into the third equation produces

$$\frac{1}{9}\left(\frac{9}{16} y^2\right) + \frac{1}{16}\, y^2 = 1 \quad \Longrightarrow \quad y^2 = 8.$$

- So, $y = \pm 2\sqrt{2}$. Because it is required that $y > 0$, choose the positive value and find that

$$x^2 = \frac{9}{16}\, y^2 = \frac{9}{16}\,(8) = \frac{9}{2} \qquad\qquad x = \frac{3}{\sqrt{2}}.$$

- So, the maximum value of f is

$$f\left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right) = 4xy = 4\left(\frac{3}{\sqrt{2}}\right)\left(2\sqrt{2}\right) = 24.$$

Example 3 (Lagrange multipliers and three variables)

Find the minimum value of

$$f(x, y, z) = 2x^2 + y^2 + 3z^2 \qquad \text{Objective function}$$

subject to the constraint $2x - 3y - 4z = 49$.

- Let $g(x, y, z) = 2x - 3y - 4z = 49$. Then, because

$$\nabla f(x, y, z) = 4x\,\mathbf{i} + 2y\,\mathbf{j} + 6z\,\mathbf{k} \quad \text{and} \quad \lambda \nabla g(x, y, z) = 2\lambda\,\mathbf{i} - 3\lambda\,\mathbf{j} - 4\lambda\,\mathbf{k}.$$

- You obtain the following system of equations.

$$\begin{aligned} 4x &= 2\lambda && f_x(x, y) = \lambda g_x(x, y) \\ 2y &= -3\lambda && f_y(x, y) = \lambda g_y(x, y) \\ 6z &= -4\lambda && f_z(x, y) = \lambda g_z(x, y) \\ 2x - 3y - 4z &= 49 && \text{Constraint} \end{aligned}$$

- The solution of this system is $x = 3$, $y = -9$, and $z = -4$. So, the optimum value of f is

$$f(3, -9, -4) = 2(3)^2 + (-9)^2 + 3(-4)^2 = 147.$$

- From the original function and constraint, it is clear that $f(x, y, z)$ has no maximum. So, the optimum value of f determined above is a minimum. ■

Example 4 (Optimization inside a region)

Find the extreme values of

$$f(x, y) = x^2 + 2y^2 - 2x + 3 \qquad \text{Objective function}$$

subject to the constraint $x^2 + y^2 \leq 10$.

- To solve this problem, you can break the constraint into two cases.
  - **a.** For points on the circle $x^2 + y^2 = 10$, you can use Lagrange multipliers to find that the maximum value of $f(x, y)$ is 24—this value occurs at $(-1, 3)$ and at $(-1, -3)$. In a similar way, you can determine that the minimum value of $f(x, y)$ is approximately 6.675—this value occurs at $(\sqrt{10}, 0)$.
  - **b.** For points inside the circle, you can use the techniques discussed in Section 13.8 to conclude that the function has a relative minimum of 2 at the point $(1, 0)$.
- By combining these two results, you can conclude that f has a maximum of 24 at $(-1, \pm 3)$ and a minimum of 2 at $(1, 0)$. ■

The method of Lagrange multipliers with two constraints

For optimization problems involving two constraint functions g and h, you can introduce a second Lagrange multiplier, μ (the lowercase Greek letter mu), and then solve the equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

where the gradient vectors are not parallel, as illustrated in Example 5.

Example 5 (Optimization with two constraints)

Let $T(x, y, z) = 20 + 2x + 2y + z^2$ represent the temperature at each point on the sphere $x^2 + y^2 + z^2 = 11$. Find the extreme temperatures on the curve formed by the intersection of the plane $x + y + z = 3$ and the sphere.

- The two constraints are

$$g(x, y, z) = x^2 + y^2 + z^2 = 11 \quad \text{and} \quad h(x, y, z) = x + y + z = 3.$$

- Using

$$\begin{aligned}
\nabla T(x,y,z) &= 2\,\mathbf{i} + 2\,\mathbf{j} + 2z\,\mathbf{k} \\
\lambda\nabla g(x,y,z) &= 2\lambda x\,\mathbf{i} + 2\lambda y\,\mathbf{j} + 2\lambda z\,\mathbf{k} \\
\mu\nabla h(x,y,z) &= \mu\,\mathbf{i} + \mu\,\mathbf{j} + \mu\,\mathbf{k}
\end{aligned}$$

you can write the following system of equations.

$$\begin{aligned}
2 &= 2\lambda x + \mu & T_x(x,y,z) &= \lambda g_x(x,y,z) + \mu h_x(x,y,z) \\
2 &= 2\lambda y + \mu & T_y(x,y,z) &= \lambda g_y(x,y,z) + \mu h_y(x,y,z) \\
2z &= 2\lambda z + \mu & T_z(x,y,z) &= \lambda g_z(x,y,z) + \mu h_z(x,y,z) \\
x^2+y^2+z^2 &= 11 & &\text{Constraint 1} \\
x+y+z &= 3 & &\text{Constraint 2}
\end{aligned}$$

- By subtracting the second equation from the first, you can obtain the following system.

$$\lambda(x-y)=0 \quad 2z(1-\lambda)-\mu=0 \quad x^2+y^2+z^2=11 \quad x+y+z=3$$

- From the first equation, you can conclude that $\lambda=0$ or $x=y$.

- If $\lambda = 0$, you can show that the critical points are $(3, -1, 1)$ and $(-1, 3, 1)$.
- If $\lambda \neq 0$, then $x = y$ and you can show that the critical points occur when $x = y = \left(3 \pm 2\sqrt{3}\right)/3$ and $z = \left(3 \mp 4\sqrt{3}\right)/3$.
- Finally, to find the optimal solutions, compare the temperatures at the four critical points.

$$
\begin{aligned}
T(3, -1, 1) = T(-1, 3, 1) &= 25 \\
T\left(\frac{3 - 2\sqrt{3}}{3}, \frac{3 - 2\sqrt{3}}{3}, \frac{3 + 4\sqrt{3}}{3}\right) &= \frac{91}{3} \approx 30.33 \\
T\left(\frac{3 + 2\sqrt{3}}{3}, \frac{3 + 2\sqrt{3}}{3}, \frac{3 - 4\sqrt{3}}{3}\right) &= \frac{91}{3} \approx 30.33
\end{aligned}
$$

- So, $T = 25$ is the minimum temperature and $T = \frac{91}{3}$ is the maximum temperature on the curve. ■