Chapter 10 Conics, Parametric Equations, and Polar Coordinates

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Plane curves and parametric equations

Definition 10.1 (Plane curve)

If f and g are continuous functions of t on an interval I, then the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are called parametric equations and t is called the parameter. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. Taken together, the parametric equations and the graph are called a plane curve, denoted by C.

- In this section, you will study situations in which three variables are used to represent a curve in the plane.

Definition 10.2 (Smooth curve)

A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called smooth if f' and g' are continuous on I and not simultaneously 0, except possibly at the endpoints of I. The curve C is called piecewise smooth if it is smooth on each subinterval of some partition of I.

Slope and tangent lines

- The projectile is represented by the parametric equations

$$x = 24\sqrt{2}t \quad \text{and} \quad y = -16t^2 + 24\sqrt{2}t$$

as shown in Figure 1. You know that these equations enable you to locate the position of the projectile at a given time.
- You also know that the object is initially projected at an angle of 45°.

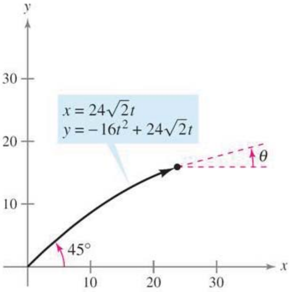

Figure 1: At time t, the angle of elevation of the projectile is θ, the slope of the tangent line at that point.

- But how can you find the angle θ representing the object's direction at some other time t?
- The following theorem answers this question by giving a formula for the slope of the tangent line as a function of t.

Theorem 10.7 (Parametric form of the derivative)

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at (x, y) is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}, \quad \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0.$$

Example 1 (Differentiation and parametric form)

Find $\mathrm{d}y/\mathrm{d}x$ for the curve given by $x = \sin t$ and $y = \cos t$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{-\sin t}{\cos t} = -\tan t$$

∎

- Because $\mathrm{d}y/\mathrm{d}x$ is a function of t, you can use Theorem 10.7 repeatedly to find higher-order derivatives.
- For instance,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]}{\mathrm{d}x/\mathrm{d}t} = \frac{-\sec^2 t}{\cos t} = -\sec^3 t$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right] = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right]}{\mathrm{d}x/\mathrm{d}t}$$

$$= \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(-\sec^3 t\right)}{\cos t} = \frac{-3\sec^2 t(\sec t \tan t)}{\cos t} = -3\sec^4 t \tan t.$$

Example 2 (Finding slope and concavity)

For the curve given by

$$x = \sqrt{t} \quad \text{and} \quad y = \frac{1}{4}(t^2 - 4), \quad t \geq 0$$

find the slope and concavity at the point $(2, 3)$.

- Because

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t} = \frac{(1/2)t}{(1/2)t^{-1/2}} = t^{3/2}$$

  you can find the second derivative to be

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}[\,\mathrm{d}y/\,\mathrm{d}x]}{\mathrm{d}x/\mathrm{d}t} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}[t^{3/2}]}{\mathrm{d}x/\,\mathrm{d}t} = \frac{(3/2)t^{1/2}}{(1/2)t^{-1/2}} = 3t.$$

- At $(x, y) = (2, 3)$, it follows that $t = 4$, and the slope is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (4)^{3/2} = 8.$$

- Moreover, when $t = 4$, the second derivative is

$$\frac{d^2y}{dx^2} = 3(4) = 12 > 0$$

and you can conclude that the graph is concave upward at $(2, 3)$, as shown in Figure 2. ■

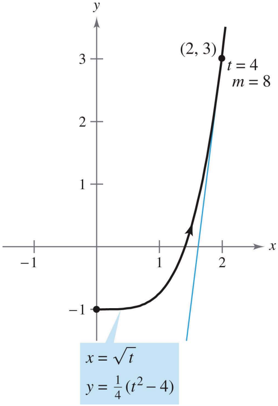

Figure 2: The graph is concave upward at $(2, 3)$, when $t = 4$.

Example 3 (A curve with two tangent lines at a point)

The prolate cycloid given by

$$x = 2t - \pi \sin t \quad \text{and} \quad y = 2 - \pi \cos t$$

crosses itself at the point $(0, 2)$, as shown in Figure 3. Find the equations of both tangent lines at this point.

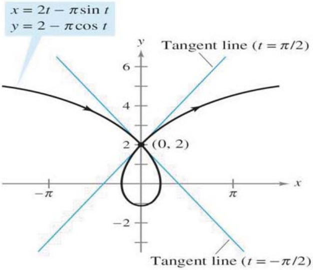

Figure 3: This prolate cycloid has two tangent lines at the point $(0, 2)$.

- Because $x = 0$ and $y = 2$ when $t = \pm\pi/2$, and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{\pi \sin t}{2 - \pi \cos t}$$

you have $\mathrm{d}y/\mathrm{d}x = -\pi/2$ when $t = -\pi/2$ and $\mathrm{d}y/\mathrm{d}x = \pi/2$ when $t = \pi/2$. So, the two tangent lines at $(0, 2)$ are

$$y - 2 = -\left(\frac{\pi}{2}\right)x \qquad \text{Tangent line when } t = -\frac{\pi}{2}$$

and

$$y - 2 = \left(\frac{\pi}{2}\right)x. \qquad \text{Tangent line when } t = \frac{\pi}{2}$$

■

Remark

If $\mathrm{d}y/\mathrm{d}t = 0$ and $\mathrm{d}x/\mathrm{d}t \neq 0$ when $t = t_0$, the curve represented by $x = f(t)$ and $y = g(t)$ has a horizontal tangent at $(f(t_0), g(t_0))$. Similarly, if $\mathrm{d}x/\mathrm{d}t = 0$ and $\mathrm{d}y/\mathrm{d}t \neq 0$ when $t = t_0$, the curve represented by $x = f(t)$ and $y = g(t)$ has a vertical tangent at $(f(t_0), g(t_0))$.

Arc length

Theorem 10.8 (Arc length in parametric form)

If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \le t \le b$ (except possibly at the endpoints), then the arc length of C over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,\mathrm{d}t.$$

- Recall from Section 7.4 that the formula for the arc length of a curve C given by $y = h(x)$ over the interval $[x_0, x_1]$ is

$$s = \int_{x_0}^{x_1} \sqrt{1 + [h'(x)]^2}\,\mathrm{d}x = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

- If C is represented by the parametric equations $x = f(t)$ and $y = g(t)$, $a \le t \le b$, and if $\mathrm{d}x/\mathrm{d}t = f'(t) > 0$, you can write

$$
\begin{aligned}
s &= \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}\right)^2}\,\mathrm{d}x \\
&= \int_a^b \sqrt{\frac{(\mathrm{d}x/\mathrm{d}t)^2 + (\mathrm{d}y/\mathrm{d}t)^2}{(\mathrm{d}x/\mathrm{d}t)^2}}\,\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\
&= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,\mathrm{d}t.
\end{aligned}
$$

$\square$

- If a circle rolls along a line, a point on its circumference will trace a path called a cycloid.
- If the circle rolls around the circumference of another circle, the path of the point is an epicycloid.

Example 4 (Finding arc length)

A circle of radius 1 rolls around the circumference of a larger circle of radius 4. The epicycloid traced by a point on the circumference of the smaller circle is given by

$$x = 5\cos t - \cos 5t \quad \text{and} \quad y = 5\sin t - \sin 5t.$$

Find the distance traveled by point in one trip about the larger circle.

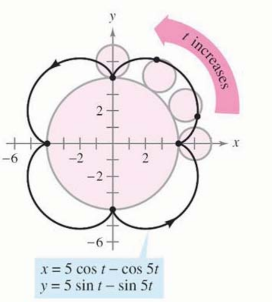

Figure 4: An epicycloid is traced by a point on the smaller circle as it rolls around the larger circle.

- Before applying Theorem 10.8, note in Figure 4 that the curve has sharp points when $t = 0$ and $t = \pi/2$.
- Between these two points, $\mathrm{d}x/\mathrm{d}t$ and $\mathrm{d}y/\mathrm{d}t$ are not simultaneously 0.
- So, the portion of the curve generated from $t = 0$ to $t = \pi/2$ is smooth.
- To find the total distance traveled by the point, you can find the arc length of that portion lying in the first quadrant and multiply by 4.

$$
\begin{aligned}
s &= 4\int_0^{\pi/2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\
&= 4\int_0^{\pi/2} \sqrt{(-5\sin t + 5\sin 5t)^2 + (5\cos t - 5\cos 5t)^2}\,\mathrm{d}t \\
&= 20\int_0^{\pi/2} \sqrt{2 - 2\sin t\sin 5t - 2\cos t\cos 5t}\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&= 20 \int_0^{\pi/2} \sqrt{2 - 2\cos 4t}\, \mathrm{d}t = 20 \int_0^{\pi/2} \sqrt{4 \sin^2 2t}\, \mathrm{d}t \\
&= 40 \int_0^{\pi/2} \sin 2t \, \mathrm{d}t = -20 \left[\cos 2t\right]_0^{\pi/2} = 40
\end{aligned}
$$

For the epicycloid shown in Figure 4, an arc length of 40 seems about right because the circumference of a circle of radius 6 is $2\pi r = 12\pi \approx 37.7$. ■

Area of a surface of revolution

Theorem 10.9 (Area of a surface of revolution)

If a smooth curve C given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \le t \le b$, then the area S of the surface of revolution formed by revolving C about the coordinate axes is given by the following.

1. $S = 2\pi \int_a^b g(t)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$
   Revolution about the x-axis: $g(t) \ge 0$
2. $S = 2\pi \int_a^b f(t)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$
   Revolution about the y-axis: $f(t) \ge 0$

- These formulas are easy to remember if you think of the differential of arc length as

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

- Then the formulas are written as follows.

  1. $S = 2\pi \int_a^b g(t)\,\mathrm{d}s$ **2.** $S = 2\pi \int_a^b f(t)\,\mathrm{d}s$

Example 5 (Finding the area of a surface of revolution)

Let C be the arc of the circle

$$x^2 + y^2 = 9$$

from $(3, 0)$ to $(3/2, 3\sqrt{3}/2)$, as shown in Figure 5. Find the area of the surface formed by revolving C about the x-axis.

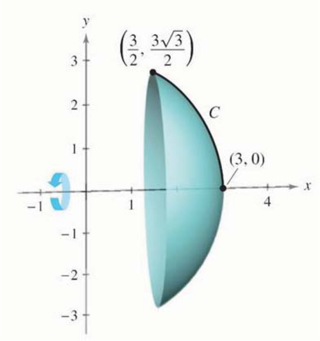

Figure 5: The surface of revolving has a surface area of 9π.

- You can represent C parametrically by the equations

$$x = 3\cos t \quad \text{and} \quad y = 3\sin t, \quad 0 \le t \le \pi/3.$$

(Note that you can determine the interval for t by observing that $t = 0$ when $x = 3$ and $t = \pi/3$ when $x = 3/2$.)

- On this interval, C is smooth and y is nonnegative, and you can apply Theorem 10.9 to obtain a surface area of

$$
\begin{aligned}
S &= 2\pi \int_0^{\pi/3} (3\sin t)\sqrt{(-3\sin t)^2 + (3\cos t)^2}\,\mathrm{d}t \\
&= 6\pi \int_0^{\pi/3} \sin t \sqrt{9(\sin^2 t + \cos^2 t)}\,\mathrm{d}t \\
&= 6\pi \int_0^{\pi/3} 3\sin t\,\mathrm{d}t = -18\pi\,[\cos t]_0^{\pi/3} = -18\pi\left(\frac{1}{2} - 1\right) = 9\pi.
\end{aligned}
$$

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Polar coordinates

- You may represent graphs as collections of points (x, y) on the rectangular coordinate system.
- The corresponding equations for these graphs have been in either rectangular or parametric form.
- In this section you will study a coordinate system called the polar coordinate system.
- To form the polar coordinate system in the plane, fix a point O, called the pole (or origin), and construct from O an initial ray called the polar axis, as shown in Figure 6.

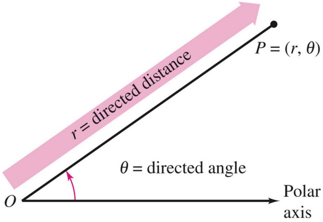


Figure 6: The definition of polar coordinates.

- Then each point P in the plane can be assigned polar coordinates (r, θ), as follows

$$r = \text{directed distance from } O \text{ to } P$$
$$\theta = \text{directed angle, counterclockwise from polar axis to segment } \overline{OP}$$

- Figure 7 shows three points on the polar coordinate system.

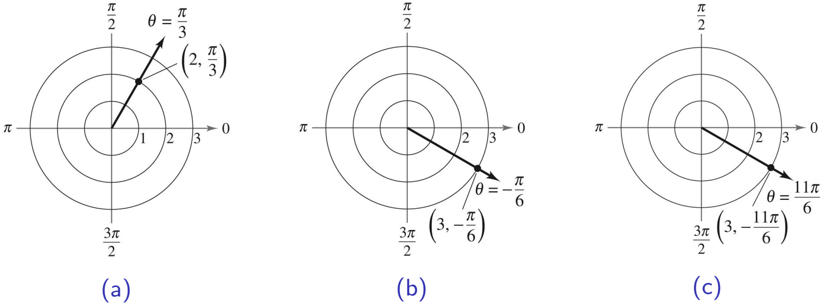


Figure 7: Points $\left(2, \frac{\pi}{3}\right)$, $\left(3, -\frac{\pi}{6}\right)$, $\left(3, \frac{11}{6}\pi\right)$ on the polar coordinates system.

- Notice that in this system, it is convenient to locate points with respect to a grid of concentric circles intersected by radial lines through the pole.
- With rectangular coordinates, each point (x, y) has a unique representation. This is not true with polar coordinates.
- For instance, the coordinates (r, θ) and $(r, 2\pi + \theta)$ represent the same point [see parts (b) and (c) in Figure 7].
- Also, because r is a directed distance, the coordinates (r, θ) and $(-r, \pi + \theta)$ represent the same point.
- In general, the point (r, θ) can be written as

$$(r, \theta) = (r, \theta + 2n\pi) \quad \text{or} \quad (r, \theta) = (-r, \theta + (2n+1)\pi)$$

where n is any integer. Moreover, the pole is represented by $(0, \theta)$, where θ is any angle.

Coordinate conversion

- To establish the relationship between polar and rectangular coordinates, let the polar axis coincide with the positive x-axis and the pole with the origin, as shown in Figure 8.
- Because (x, y) lies on a circle of radius r, it follows that $r^2 = x^2 + y^2$. Moreover, for $r > 0$ the definitions of the trigonometric functions imply that $\tan\theta = \frac{y}{x}$, $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$.
- If $r < 0$, you can show that the same relationships hold.

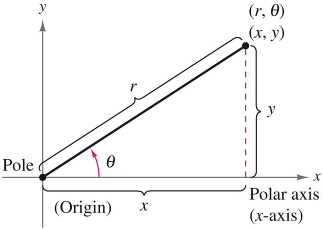


Figure 8: Relating polar and rectangular coordinates.

Theorem 10.10 (Polar-to-rectangular conversion)

The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of the point as follows.
1. $x = r\cos\theta$ *and* $y = r\sin\theta$. **2.** $\tan\theta = \frac{y}{x}$ *and* $r^2 = x^2 + y^2$.

Example 1 (Polar-to-rectangular conversion)

a. For the point $(r, \theta) = (2, \pi)$, $x = r\cos\theta = 2\cos\pi = -2$ and $y = r\sin\theta = 2\sin\pi = 0$.
So, the rectangular coordinates are $(x, y) = (-2, 0)$.

b. For the point $(r, \theta) = (\sqrt{3}, \pi/6)$, $x = \sqrt{3}\cos\frac{\pi}{6} = \frac{3}{2}$ and $y = \sqrt{3}\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.
So, the rectangular coordinates are $(x, y) = (3/2, \sqrt{3}/2)$. See Figure 9. ■

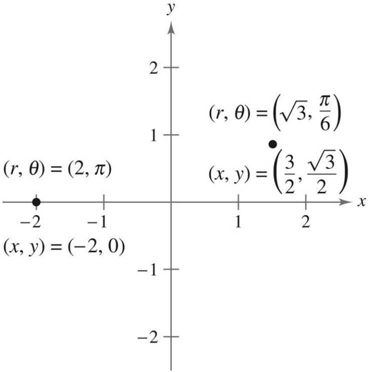


Figure 9: To convert from polar to rectangular coordinates, let $x = r\cos\theta$ and $y = r\sin\theta$.

Example 2 (Rectangular-to-polar conversion)

a. For the second quadrant point $(x, y) = (-1, 1)$,

$$\tan\theta = \frac{y}{x} = -1 \implies \theta = \frac{3\pi}{4}.$$

Because θ was chosen to be in the same quadrant as (x, y), you should use a positive value of r.

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

This implies that one set of polar coordinates is $(r, \theta) = (\sqrt{2}, 3\pi/4)$.

b. Because the point $(x, y) = (0, 2)$ lies on the positive y-axis, choose $\theta = \pi/2$ and $r = 2$, and one set of polar coordinates is $(r, \theta) = (2, \pi/2)$. See Figure 10. ■

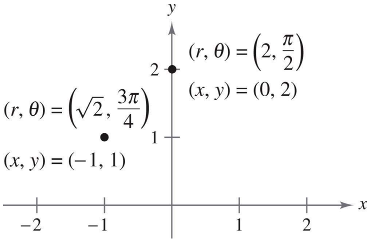


Figure 10: To convert from rectangular to polar coordinates, let $\tan \theta = y/x$ and $r = \sqrt{x^2 + y^2}$.

Polar graphs

Example 3 (Graphing polar equations)

Describe the graph of each polar equation. Confirm each description by converting to a rectangular equation.

a. $r = 2$ **b.** $\theta = \frac{\pi}{3}$ **c.** $r = \sec\theta$

a. The graph of the polar equation $r = 2$ consists of all points that are two units from the pole. In other words, this graph is a circle centered at the origin with a radius of 2. [See Figure 11(a).]

- You can confirm this by using the relationship $r^2 = x^2 + y^2$ to obtain the rectangular equation $x^2 + y^2 = 2^2$.

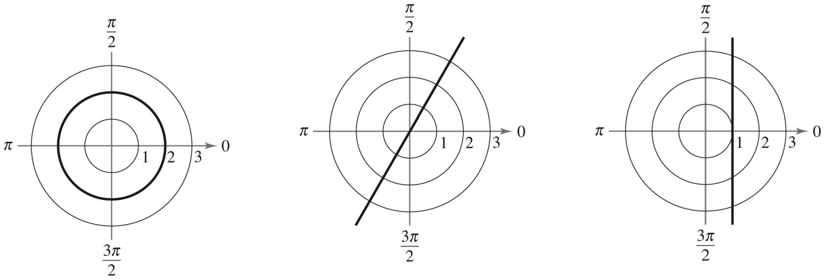
b. The graph of the polar equation $\theta = \pi/3$ consists of all points on the line that makes an angle of $\pi/3$ with the positive x-axis. [See Figure 11(b).]

- You can confirm this by using the relationship $\tan\theta = y/x$ to obtain the rectangular equation $y = \sqrt{3}x$.

c. The graph of the polar equation $r = \sec\theta$ is not evident by simple inspection, so you can begin by converting to rectangular form using the relationship $r\cos\theta = x$.

$$r = \sec\theta \qquad\qquad r\cos\theta = 1 \qquad\qquad x = 1$$

- From the rectangular equation, you can see that the graph is a vertical line. [See Figure 11(c).] ■



(a) Circle: $r = 2$.

(b) Radial line: $\theta = \frac{\pi}{3}$.

(c) Vertical line: $r = \sec\theta$.

Figure 11: Graphing polar equations.

- The graph of $r = \frac{1}{2}\,\theta$ shown in Figure 12 was produced with a graphing calculator in parametric mode. This equation was graphed using the parametric equations $x = \frac{1}{2}\,\theta\cos\theta$ and $y = \frac{1}{2}\,\theta\sin\theta$ with the values of θ varying from -4π to 4π.
- This curve is of the form $r = a\theta$ and is called a spiral of Archimedes.

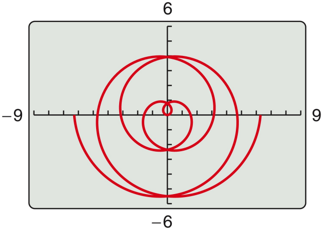


Figure 12: Spiral of Archimedes.

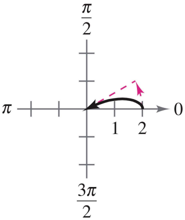
Example 4 (Sketching a polar graph)

Sketch the graph of $r = 2\cos 3\theta$.

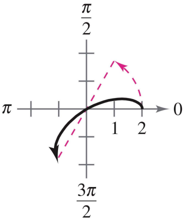
- Begin by writing the polar equation in parametric form.

$$x = 2\cos 3\theta \cos\theta \quad \text{and} \quad y = 2\cos 3\theta \sin\theta$$

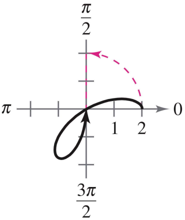
- After some experimentation, you will find that the entire curve, which is called a rose curve, can be sketched by letting θ vary from 0 to π, as shown in Figure 13. If you try duplicating this graph with a graphing utility, you will find that by letting θ vary from 0 to 2π, you will actually trace the entire curve twice. ■



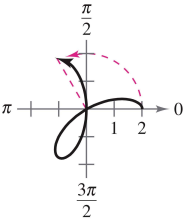
(a) $0 \le \theta \le \pi/6$.



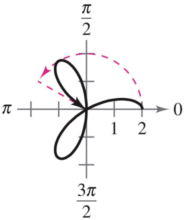
(b) $0 \le \theta \le \pi/3$.



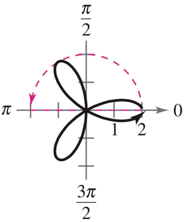
(c) $0 \le \theta \le \pi/2$.



(d) $0 \le \theta \le 2\pi/3$.



(e) $0 \le \theta \le 5\pi/6$.



(f) $0 \le \theta \le \pi$.

Figure 13: Sketching a polar graph.

Slope and tangent lines

- To find the slope of a tangent line to a polar graph, consider a differentiable function given by $r = f(\theta)$. To find the slope in polar form, use the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = f(\theta)\sin\theta.$$

- Using the parametric form of $\mathrm{d}y/\mathrm{d}x$ given in Theorem 10.7, you have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$

Theorem 10.11 (Slope in polar form)

If f is a differentiable function of θ, then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

provided that $\mathrm{d}x/\mathrm{d}\theta \neq 0$ at (r, θ). (See Figure 14.)

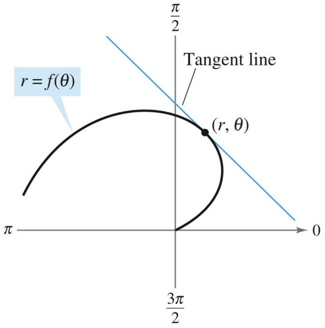


Figure 14: Tangent line to polar curve $r = f(\theta)$.

- From Theorem 10.11, you can make the following observations.
  1. Solution to $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0$ yield horizontal tangents, provided that $\frac{\mathrm{d}x}{\mathrm{d}\theta} \neq 0$.
  2. Solution to $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 0$ yield vertical tangents, provided that $\frac{\mathrm{d}y}{\mathrm{d}\theta} \neq 0$.
- If $\mathrm{d}y/\mathrm{d}\theta$ and $\mathrm{d}x/\mathrm{d}\theta$ are simultaneously 0, no conclusion can be drawn about tangent lines.

Example 5 (Finding horizontal and vertical tangent lines)

Find the horizontal and vertical tangent lines of $r = \sin\theta$, $0 \le \theta \le \pi$.

- Begin by writing the equation in parametric form.

$$x = r\cos\theta = \sin\theta\cos\theta \quad \text{and} \quad y = r\sin\theta = \sin^2\theta$$

- Next, differentiate x and y with respect to θ and set each derivative equal to 0.

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \cos^2\theta - \sin^2\theta = \cos 2\theta = 0 \quad \Longrightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 2\sin\theta\cos\theta = \sin 2\theta = 0 \quad \Longrightarrow \quad \theta = 0, \frac{\pi}{2}$$

- So, the graph has vertical tangent lines at $\left(\sqrt{2}/2, \pi/4\right)$ and $(\sqrt{2}/2, 3\pi/4)$, and it has horizontal tangent lines at $(0, 0)$ and $(1, \pi/2)$, as shown in Figure 15. ■

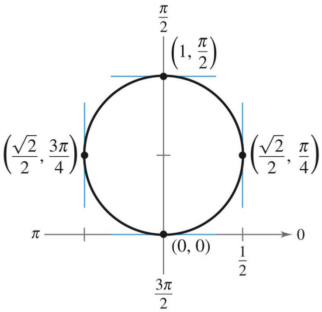


Figure 15: Horizontal and vertical tangent lines of $r = \sin\theta$.

Example 6 (Finding horizontal and vertical tangent lines)

Find the horizontal and vertical tangents to the graph of $r = 2(1 - \cos\theta)$.

- Using $y = r\sin\theta$, differentiate and set $\mathrm{d}y/\mathrm{d}\theta$ equal to 0.

$$\begin{aligned} y &= r\sin\theta = 2(1-\cos\theta)\sin\theta \\ \frac{\mathrm{d}y}{\mathrm{d}\theta} &= 2[(1-\cos\theta)(\cos\theta) + \sin\theta(\sin\theta)] \\ &= -2(2\cos\theta + 1)(\cos\theta - 1) = 0 \end{aligned}$$

- So, $\cos\theta = -\frac{1}{2}$ and $\cos\theta = 1$, and you can conclude that $\mathrm{d}y/\mathrm{d}\theta = 0$ when $\theta = 2\pi/3$, $4\pi/3$ and 0.
- Similarly, using $x = r\cos\theta$, you have

$$\begin{aligned} x &= r\cos\theta = 2\cos\theta - 2\cos^2\theta \\ \frac{\mathrm{d}x}{\mathrm{d}\theta} &= -2\sin\theta + 4\cos\theta\sin\theta = 2\sin\theta(2\cos\theta - 1) = 0. \end{aligned}$$

- So, $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$, and you can conclude that $\mathrm{d}x/\mathrm{d}\theta = 0$ when $\theta = 0$, π, $\pi/3$, $5\pi/3$.

- From these results, and from the graph shown in Figure 16, you can conclude that the graph has horizontal tangents at $(3, 2\pi/3)$ and $(3, 4\pi/3)$, and has vertical tangents at $(1, \pi/3)$, $(1, 5\pi/3)$, and $(4, \pi)$. ■

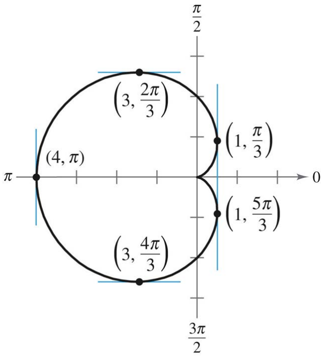


Figure 16: Horizontal and vertical tangent lines of $r = 2(1 - \cos\theta)$.

- This graph is called a cardioid.
- Note that both derivatives ($\mathrm{d}y/\mathrm{d}\theta$ and $\mathrm{d}x/\mathrm{d}\theta$) are 0 when $\theta = 0$.
- Using this information alone, you don't know whether the graph has a horizontal or vertical tangent line at the pole. From Figure 16, however, you can see that the graph has a cusp at the pole.

Theorem 10.12 (Tangent lines at the pole)

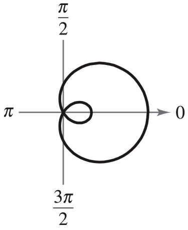
If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the line $\theta = \alpha$ is tangent at the pole to the graph of $r = f(\theta)$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(\alpha)\cos\alpha + f'(\alpha)\sin\alpha}{-f(\alpha)\sin\alpha + f'(\alpha)\cos\alpha} = \frac{0 + f'(\alpha)\sin\alpha}{0 + f'(\alpha)\cos\alpha} = \tan\alpha$$

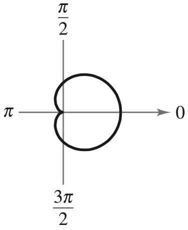
So, the line $\theta = \alpha$ is the tangent to the graph at the pole, $(0, \alpha)$. $\square$

Special polar graphs

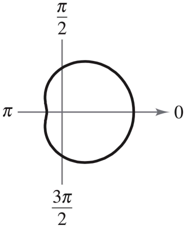
- Several important types of graphs have equations that are simpler in polar form than in rectangular form.
- For example, the polar equation of a circle having a radius of a and centered at the origin is simply $r = a$. Several other types of graphs that have simpler equations in polar form are shown below.



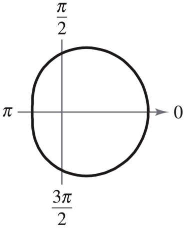
(a) $\frac{a}{b} < 1$. Limacon with inner loop.



(b) $\frac{a}{b} = 1$. Cardioid (heart-shaped).

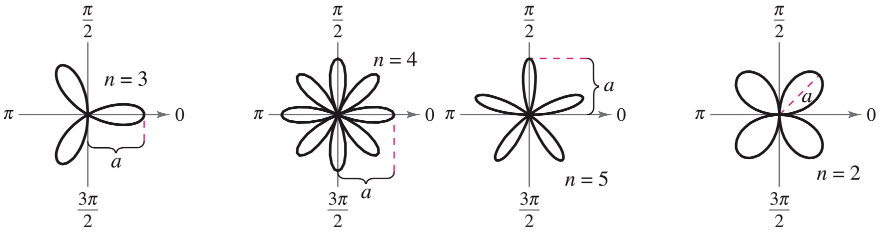


(c) $1 < \frac{a}{b} < 2$. Dimpled limaçon.



(d) $\frac{a}{b} \geq 2$. Convex limaçon.

Figure 17: Limacon: $r = a \pm b\cos\theta$, $r = a \pm b\sin\theta$ $(a > 0, b > 0)$.



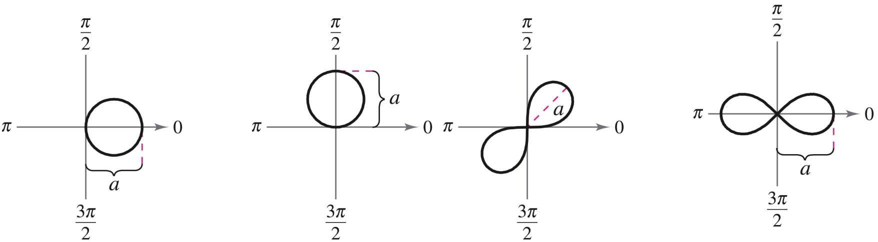
(a) $r = a \cos n\theta$. Rose curve.

(b) $r = a \cos n\theta$. Rose curve.

(c) $r = a \sin n\theta$. Rose curve.

(d) $r = a \sin n\theta$. Rose curve.

Figure 18: Rose curves: n petals if n is odd, $2n$ petals if n is even ($n \geq 2$).



(a) $r = a\cos\theta$. Circle.

(b) $r = a\sin\theta$. Circle.

(c) $r^2 = a^2\sin 2\theta$. Lemniscate.

(d) $r^2 = a^2\cos 2\theta$. Lemniscate.

Figure 19: Circles and Lemniscate.

Table of Contents

Area of a polar region

- The development of a formula for the area of a polar region parallels that for the area of a region on the rectangular coordinate system, but uses sectors of a circle instead of rectangles as the basic elements of area.
- In Figure 20, note that the area of a circular sector of radius r is given by $\frac{1}{2}\theta r^2$ provided θ is measured in radians.

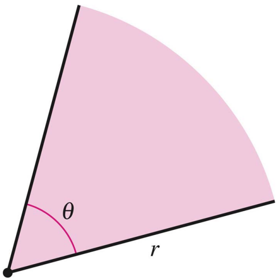


Figure 20: The area of a sector of a circle is $A = \frac{1}{2}\theta r^2$.

- Consider the function given by $r = f(\theta)$, where f is continuous and nonnegative on the interval given by $\alpha \leq \theta \leq \beta$. The region bounded by the graph of f and the radial lines $\theta = \alpha$ and $\theta = \beta$ is shown in Figure 21.

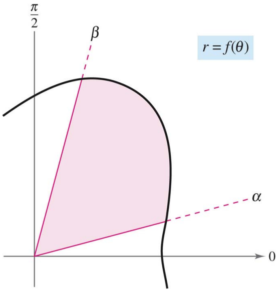


Figure 21: Area in polar coordinates.

- To find the area of this region, partition the interval $[\alpha, \beta]$ into n equal subintervals

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \theta_n = \beta$$

- Then approximate the area of the region by the sum of the areas of the n sectors, as shown in Figure 21.
- Radius of ith sector $= f(\theta_i)$.
- Central angle of ith sector $= \frac{\beta - \alpha}{n} = \triangle\theta$

$$A \approx \sum_{i=1}^{n} \left(\frac{1}{2}\right) \triangle\theta [f(\theta_i)]^2.$$

- Taking the limit as $n \to \infty$ produces

$$A = \lim_{n\to\infty} \frac{1}{2} \sum_{i=1}^{n} [f(\theta_i)]^2 \triangle\theta = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, \mathrm{d}\theta.$$

Theorem 10.13 (Area in polar coordinates)

If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \le 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, \mathrm{d}\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, \mathrm{d}\theta, \quad 0 < \beta - \alpha \le 2\pi.$$

Example 1 (Finding the area of a polar region)

Find the area of one petal of the rose curve given by $r = 3\cos 3\theta$.

- In Figure 22, you can see that the petal on the right is traced as θ increases from $-\pi/6$ to $\pi/6$.
- So, the area is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,\mathrm{d}\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} (3\cos 3\theta)^2\,\mathrm{d}\theta = \frac{9}{2}\int_{-\pi/6}^{\pi/6} \frac{1+\cos 6\theta}{2}\,\mathrm{d}\theta$$

$$= \frac{9}{4}\left[\theta + \frac{\sin 6\theta}{6}\right]_{-\pi/6}^{\pi/6} = \frac{9}{4}\left(\frac{\pi}{6}+\frac{\pi}{6}\right) = \frac{3\pi}{4}. \quad \blacksquare$$

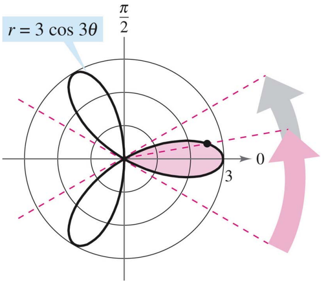


Figure 22: The area of one petal of the rose curve that lies between the radial lines $\theta = -\pi/6$ and $\theta = \pi/6$ is $3\pi/4$.

Example 2 (Finding the area bounded by a single curve)

Find the area of the region lying between the inner and outer loops of the limacon $r = 1 - 2\sin\theta$.

- In Figure 23, note that the inner loop is traced as θ increases from $\pi/6$ to $5\pi/6$. So, the area inside the inner loop is

$$\begin{aligned}
A_1 &= \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1 - 2\sin\theta)^2 \, d\theta \\
&= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(1 - 4\sin\theta + 4\sin^2\theta\right) d\theta \\
&= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[1 - 4\sin\theta + 4\left(\frac{1-\cos 2\theta}{2}\right)\right] d\theta \\
&= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3 - 4\sin\theta - 2\cos 2\theta) \, d\theta \\
&= \frac{1}{2}\left[3\theta + 4\cos\theta - \sin 2\theta\right]_{\pi/6}^{5\pi/6} = \frac{1}{2}\left(2\pi - 3\sqrt{3}\right) = \pi - \frac{3\sqrt{3}}{2}.
\end{aligned}$$

- In a similar way, you can integrate from $5\pi/6$ to $13\pi/6$ to find that the area of the region lying inside the outer loop is $A_2 = 2\pi + (3\sqrt{3}/2)$. The area of the region lying between the two loops is the difference of A_2 and A_1.

$$A = A_2 - A_1 = \left(2\pi + \frac{3\sqrt{3}}{2}\right) - \left(\pi - \frac{3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3} \approx 8.34$$

■

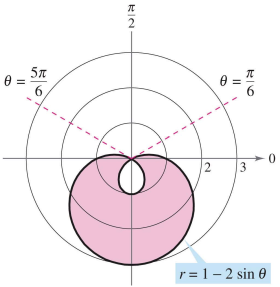


Figure 23: The area between the inner and outer loops is approximately 8.34.

Points of intersection of polar graphs

- Because a point may be represented in different ways in polar coordinates, care must be taken in determining the points of intersection of two polar graphs.
- For example, consider the points of intersection of the graphs of $r = 1 - 2\cos\theta$ and $r = 1$ as shown in Figure 24.

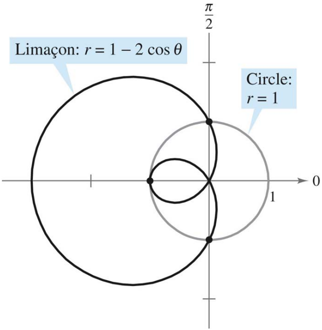


Figure 24: Limacon: $r = 1 - 2\cos\theta$ and three points intersection: $(1, \pi/2)$, $(-1, 0)$, $(1, 3\pi/2)$.

- If, as with rectangular equations, you attempted to find the points of intersection by solving the two equations simultaneously, you would obtain

$$r = 1 - 2\cos\theta \qquad 1 = 1 - 2\cos\theta \qquad \cos\theta = 0 \qquad \theta = \frac{\pi}{2}, \ \frac{3\pi}{2}$$

- The corresponding points of intersection are $(1, \pi/2)$ and $(1, 3\pi/2)$.
- However, from Figure 24 you can see that there is a third point of intersection that did not show up when the two polar equations were solved simultaneously.
- The reason the third point was not found is that it does not occur with the same coordinates in the two graphs.
- On the graph of $r = 1$, the point occurs with coordinates $(1, \pi)$, but on the graph of $r = 1 - 2\cos\theta$, the point occurs with coordinates $(-1, 0)$.

- You can compare the problem of finding points of intersection of two polar graphs with that of finding collision points of two satellites in intersecting orbits about Earth, as shown in Figure 25.

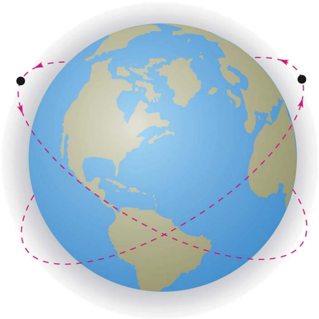


Figure 25: The paths of satellites can cross without causing a collision.

Example 3 (Finding the area of a region between two curves)

Find the area of the region common to the two regions bounded by the following curves.

$$r = -6\cos\theta \qquad \text{Circle} \qquad\qquad r = 2 - 2\cos\theta \qquad \text{Cardioid}$$

- Because both curves are symmetric with respect to the x-axis, you can work with the upper half-plane, as shown in Figure 26.

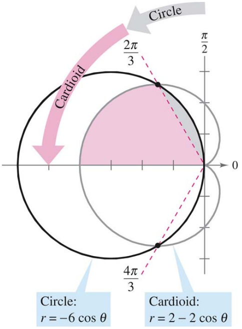


Figure 26: Find the area between circle $r = -6\cos\theta$ and cardioid $r = 2 - 2\cos\theta$.

- The gray shaded region lies between the circle and the radial line $\theta = 2\pi/3$.
- Because the circle has coordinates $(0, \pi/2)$ at the pole, you can integrate between $\pi/2$ and $2\pi/3$ to obtain the area of this region.
- The region that is shaded red is bounded by the radial lines $\theta = 2\pi/3$ and $\theta = \pi$ and the cardioid.
- So, you can find the area of this second region by integrating between $2\pi/3$ and π.
- The sum of these two integrals gives the area of the common region lying above the radial line $\theta = \pi$.

$$
\begin{aligned}
\frac{A}{2} &= \overbrace{\frac{1}{2}\int_{\pi/2}^{2\pi/3}(-6\cos\theta)^2\,\mathrm{d}\theta}^{\text{Region between circle and radial line } \theta = 2\pi/3} + \overbrace{\frac{1}{2}\int_{2\pi/3}^{\pi}(2-2\cos\theta)^2\,\mathrm{d}\theta}^{\text{Region between cardioid and radial lines } \theta = 2\pi/3 \text{ and } \theta = \pi} \\
&= 18\int_{\pi/2}^{2\pi/3}\cos^2\theta\,\mathrm{d}\theta + \frac{1}{2}\int_{2\pi/3}^{\pi}(4-8\cos\theta+4\cos^2\theta)\,\mathrm{d}\theta \\
&= 9\int_{\pi/2}^{2\pi/3}(1+\cos 2\theta)\,\mathrm{d}\theta + \int_{2\pi/3}^{\pi}(3-4\cos\theta+\cos 2\theta)\,\mathrm{d}\theta \\
&= 9\left[\theta+\frac{\sin 2\theta}{2}\right]_{\pi/2}^{2\pi/3} + \left[3\theta-4\sin\theta+\frac{\sin 2\theta}{2}\right]_{2\pi/3}^{\pi} \\
&= 9\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{4}-\frac{\pi}{2}\right) + \left(3\pi-2\pi+2\sqrt{3}+\frac{\sqrt{3}}{4}\right) = \frac{5\pi}{2} \approx 7.85
\end{aligned}
$$

- Finally, multiplying by 2, you can conclude that the total area is $5\pi \approx 15.7$.

Arc length in polar form

- The formula for the length of a polar arc can be obtained from the arc length formula for a curve described by parametric equations.

Theorem 10.14 (Arc length of a polar curve)

Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, \mathrm{d}\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\, \mathrm{d}\theta.$$

- In parametric form,

$$s = \int_{a}^{b} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t.$$

- Using θ instead of t, you have $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$. So,

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta.$$

- It follows that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2.$$

- So,

$$s = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,\mathrm{d}\theta.$$

□

Example 4 (Finding the length of a polar curve)

Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the cardioid $r = f(\theta) = 2 - 2\cos\theta$ as shown in Figure 27.

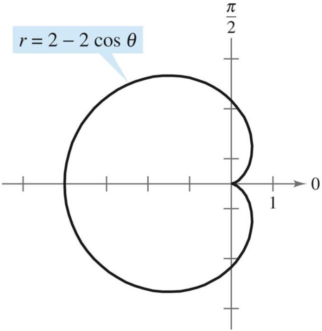


Figure 27: The cardioid $r = 2 - 2\cos\theta$.

- Because $f'(\theta) = 2\sin\theta$, you can find the arc length as follows.

$$s = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,\mathrm{d}\theta = \int_0^{2\pi} \sqrt{(2 - 2\cos\theta)^2 + (2\sin\theta)^2}\,\mathrm{d}\theta$$

$$= 2\sqrt{2}\int_0^{2\pi} \sqrt{1 - \cos\theta}\,\mathrm{d}\theta = 2\sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2\frac{\theta}{2}}\,\mathrm{d}\theta$$

$$= 4\int_0^{2\pi} \sin\frac{\theta}{2}\,\mathrm{d}\theta = 8\left[-\cos\frac{\theta}{2}\right]_0^{2\pi} = 8(1+1) = 16$$

- In the fifth step of the solution, it is legitimate to write

$$\sqrt{2\sin^2(\theta/2)} = \sqrt{2}\sin(\theta/2) \quad \text{rather than}$$

$$\sqrt{2\sin^2(\theta/2)} = \sqrt{2}|\sin(\theta/2)|$$

because $\sin(\theta/2) \geq 0$ for $0 \leq \theta \leq 2\pi$. ■

Area of a surface of revolution

- The polar coordinate versions of the formulas for the area of a surface of revolution can be obtained from the parametric versions, using the equations $x = r\cos\theta$ and $y = r\sin\theta$.

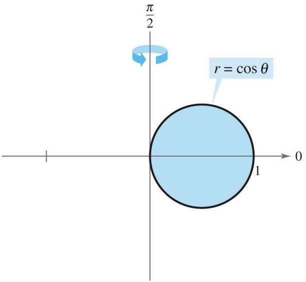
Theorem 10.15 (Area of a surface of revolution)

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line as follows.

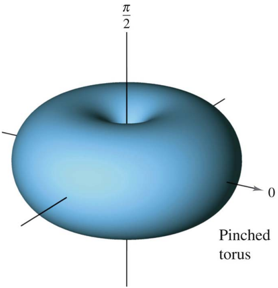
1. $S = 2\pi \int_\alpha^\beta y\, \mathrm{d}s = 2\pi \int_\alpha^\beta f(\theta)\sin\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, \mathrm{d}\theta$
   About the polar axis
2. $S = 2\pi \int_\alpha^\beta x\, \mathrm{d}s = 2\pi \int_\alpha^\beta f(\theta)\cos\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, \mathrm{d}\theta$
   About the line $\theta = \frac{\pi}{2}$

Example 5 (Finding the area of a surface of revolution)

Find the area of the surface formed by revolving the circle $r = f(\theta) = \cos\theta$ about the line $\theta = \pi/2$, as shown in Figure 28.



(a) Circle.



(b) Pinched torus.

Figure 28: Revolving a circle $r = \cos\theta$ around $x = \frac{\pi}{2}$.

- You can use the second formula given in Theorem 10.15 with $f'(\theta) = -\sin\theta$. Because the circle is traced once as θ increases from 0 to π, you have

$$
\begin{aligned}
S &= 2\pi \int_{\alpha}^{\beta} f(\theta)\cos\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, \mathrm{d}\theta \\
&= 2\pi \int_0^{\pi} \cos\theta(\cos\theta)\sqrt{\cos^2\theta + \sin^2\theta}\, \mathrm{d}\theta = 2\pi \int_0^{\pi} \cos^2\, \mathrm{d}\theta \\
&= \pi \int_0^{\pi} (1 + \cos 2\theta)\, \mathrm{d}\theta = \pi \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi} = \pi^2.
\end{aligned}
$$

■