1. (20%) Determine whether the series converges absolutely or conditionally, or diverges. In addition, please indicate the test you use.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n(2n-1)}{3n+4}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^3+1}+\sqrt{n^3}}$

(c) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n+1}$

(d) $\sum_{n=3}^{\infty} \frac{(-1)^n}{n(\ln n)[\ln(\ln n)]^2}$

(e) $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)$

Theorem 9.1 (Limit of a sequence)

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x\to\infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n\to\infty} a_n = L.$$

Definition 9.4 (Convergent and divergent series)

For the infinite series $\sum_{n=1}^{\infty} a_n$ the nth partial sum is given by

$$S_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ converges. The limit S is called the sum of the series.

$$S = a_1 + a_2 + \cdots + a_n + \cdots \qquad S = \sum_{n=1}^{\infty} a_n$$

If $\{S_n\}$ diverges, then the series diverges.

Definition 9.5 (Absolute and conditional convergence)

1. $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.
2. $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

SUMMARY OF TESTS FOR SERIES

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series ($r \neq 0$)	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	
Alternating Series ($a_n > 0$)	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\lvert R_N\rvert \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)\,dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$ or $= \infty$	Test is inconclusive when $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert > 1$ or $= \infty$	Test is inconclusive when $\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

Ans:

(a) Diverges because of $\lim_{n\to\infty} \frac{(-1)^n(2n-1)}{3n+4} = \frac{2}{3}\lim_{n\to\infty} (-1)^n \neq 0$ (by the n-th term test)

(b) $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^3}}}{\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}} = 2$, since $\frac{1}{\sqrt{n^3}}$ is a p-series with $p > 1$ which means that

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}$ is convergent by the limit comparison test. Therefore,

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^3+1}+\sqrt{n^3}}$ is absolute convergent.

(c) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n+1}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}$. The given series converges by the alternating series test since $\frac{1}{(n+1)+1}<\frac{1}{n+1}$ and $\lim_{n\to\infty}\frac{1}{n+1}=0$, but $\sum_{n=1}^{\infty}\left|\frac{\cos(n\pi)}{n+1}\right|=\sum_{n=1}^{\infty}\frac{1}{n+1}$ diverges by a limit comparison test to the divergent harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$.

Therefore, the series converges conditionally.

(d) It is absolute convergent because of $\sum_{n=3}^{\infty}\left|\frac{(-1)^n}{n(\ln n)[\ln(\ln n)]^2}\right|=\sum_{n=3}^{\infty}\frac{1}{n(\ln n)[\ln(\ln n)]^2}$. Let $f(x)=\frac{1}{x(\ln x)[\ln(\ln x)]^2}$, since $\frac{1}{(x+1)(\ln(x+1))[\ln(\ln(x+1))]^2}<\frac{1}{(x)(\ln(x))[\ln(\ln(x))]^2}$. $f(x)$ is positive, continuous and decreasing.

And $\int_3^{\infty}\frac{1}{x(\ln x)[\ln(\ln x)]^2}dx=\lim_{b\to\infty}\frac{-1}{\ln(\ln x)}\Big|_3^b<\infty$ (by integral test, it is convergent)

(e) $n^{\frac{1}{n}}-1=e^{\frac{\ln n}{n}}-1=\left[1+\frac{\ln n}{n}+\frac{1}{2}\left(\frac{\ln n}{n}\right)^2+\cdots\right]-1=\frac{\ln n}{n}+\frac{1}{2}\left(\frac{\ln n}{n}\right)^2+\cdots>\frac{\ln n}{n}>\frac{1}{n}$ $(n\geq 3)$. By the direct comparison test, the series diverges.

POWER SERIES FOR ELEMENTARY FUNCTIONS

Function	Interval of Convergence
$\frac{1}{x}=1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4-\cdots+(-1)^n(x-1)^n+\cdots$	$0<x<2$
$\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5+\cdots+(-1)^n x^n+\cdots$	$-1<x<1$
$\ln x=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots+\frac{(-1)^{n-1}(x-1)^n}{n}+\cdots$	$0<x\leq 2$
$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots+\frac{x^n}{n!}+\cdots$	$-\infty<x<\infty$
$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots$	$-\infty<x<\infty$
$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots$	$-\infty<x<\infty$
$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots+\frac{(-1)^n x^{2n+1}}{2n+1}+\cdots$	$-1\leq x\leq 1$
$\arcsin x=x+\frac{x^3}{2\cdot 3}+\frac{1\cdot 3x^5}{2\cdot 4\cdot 5}+\frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7}+\cdots+\frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)}+\cdots$	$-1\leq x\leq 1$
$(1+x)^k=1+kx+\frac{k(k-1)x^2}{2!}+\frac{k(k-1)(k-2)x^3}{3!}+\cdots+\frac{k(k-1)\cdots(k-n+1)x^n}{n!}+\cdots$	$-1<x<1$*

* The convergence at $x=\pm 1$ depends on the value of k.

Theorem 9.19 (Taylor's Theorem)

If a function f is differentiable through order $n+1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

Definition 9.7 (Power series)

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a power series. More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots$$

is called a power series centered at c, where c is a constant.

Theorem 9.20 (Convergence of a power series)

For a power series centered at c, precisely one of the following is true.

1. *The series converges only at c.*
2. *There exists a real number $R > 0$ such that the series converges absolutely for $|x-c| < R$, and diverges for $|x-c| > R$.*
3. *The series converges absolutely for all x.*

The number R is the radius of convergence. If the series converges only at c, the radius of convergence is $R = 0$, and if the series converges for all x, the radius of convergence is $R = \infty$. The set of all values of x for which it converges is the interval of convergence of the power series.

Theorem 9.21 (Properties of functions defined by power series)

If the function given by
$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$
has a radius of convergence of $R > 0$, then, on the interval $(c-R,\ c+R)$, f is differentiable (and therefore continuous).
Moreover, the derivative and antiderivative of f are as follows.

1. $f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$

2.
$\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} = C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots$

The radius of convergence of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The interval of convergence, however, may differ as a result of the behavior at the endpoints.

2. (8%) Consider the function.

$$f(x) = \frac{1}{2x-1}, x \neq \frac{1}{2}$$

(a) Find the power series expansion of $p(x)$ of f expand at the point $\frac{1}{3}$ and determine its interval of convergence.

(b) Write $p(x) = \sum_{n=0}^{\infty} a_n (x-\frac{1}{3})^n$. Is $\sum_{n=0}^{\infty} a_n (\frac{2}{3})^n = f(1) = 1$? and why?

Ans:

(a) $f(x) = \frac{-1}{1-2x} = \frac{-1}{\frac{1}{3}-2(x-\frac{1}{3})} = \frac{-3}{1-6(x-\frac{1}{3})}$

Using the geometric series allows us to obtain

$$p(x) = -3\sum_{n=0}^{\infty}\left(6\left(x-\frac{1}{3}\right)\right)^n = \sum_{n=0}^{\infty}(-3)6^n\left(x-\frac{1}{3}\right)^n,$$

which converges when $\left|6(x-\frac{1}{3})\right| < 1$ or equivalently, the interval of convergence of the power series p is $\frac{1}{6} < x < \frac{1}{2}$

(b) No, because the point 1 does not belong to the interval of convergence of p.

> **Definition 9.8 (Taylor and Maclaurin series)**
>
> If a function f has derivatives of all orders at $x = c$, then the series
>
> $$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$
>
> is called the Taylor series for $f(x)$ at c. Moreover, if $c = 0$, then the series is the Maclaurin series for f.

3. (10%)

(a) Find the Maclaurin series for $\arccos(x)$

(b) Find the radius and interval of convergence of the Maclaurin series for $\arccos(x)$. (You can ignore examine about the endpoint of the interval)

Ans:

(a) $\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}} = -\sum_{n=0}^{\infty}\binom{\frac{-1}{2}}{n}(-x^2)^n$

$$\arccos(x) = \sum_{n=0}^{\infty}\binom{\frac{-1}{2}}{n}(-1)^{n+1}\frac{x^{2n+1}}{2n+1} + C$$

Substitute 0 into the equation we have $C = \frac{\pi}{2}$. Therefore,

$$\arccos(x) = \frac{\pi}{2} + \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{n} (-1)^{n+1} \frac{x^{2n+1}}{2n+1}$$

(b) Use ratio test, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\binom{\frac{-1}{2}}{n+1}(-1)^{n+2}\frac{x^{2n+3}}{2n+3}}{\binom{\frac{-1}{2}}{n}(-1)^{n+1}\frac{x^{2n+1}}{2n+1}} \right| = \lim_{n\to\infty} \left|\frac{(n+\frac{1}{2})(2n+1)}{(n+1)(2n+3)}\right| x^2 = x^2$

When $x^2 < 1$, it is converge, therefore $\mathrm{R} = 1$ and the interval of convergence is $[-1,1]$. For the boundary at 1 and -1, see https://proofwiki.org/wiki/Power_Series_Expansion_for_Real_Arccosine_Function for more details.

4. (8%) Use a power series to approximate $\int_0^1 \sin(x^2)dx$ with an error of less than 0.001

Ans: $\int_0^1 \sin(x^2)dx = \int_0^1 (x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \cdots)dx = \left(\frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320} - \cdots\right)\Big|_0^1 = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \cdots$

It is an alternating series, therefore we know that $\int_0^1 \sin(x^2)dx = \frac{1}{3} - \frac{1}{42} \approx \frac{13}{42}$ and the error is smaller than $\frac{1}{1320}$ which is also smaller than $\frac{1}{1000}$

5. (9%) Evaluate the following expression (Try to use the Basic series of Taylor series and notice that the power series is a continuous function)

(a) $1 - \frac{\pi^2}{4^2 \times 2!} + \frac{\pi^4}{4^4 \times 4!} - \frac{\pi^6}{4^6 \times 6!} + \cdots$

(b) $\frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5(\sqrt{3})^5} - \frac{1}{7(\sqrt{3})^7} + \cdots$

(c) $\lim_{x\to 0} \frac{\tan(x) - \sin(x)}{x^2}$

Ans:

(a) $1 - \frac{\pi^2}{4^2 \times 2!} + \frac{\pi^4}{4^4 \times 4!} - \frac{\pi^6}{4^6 \times 6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \left(\frac{\pi}{4}\right)^{2n} = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

(b) $\frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5(\sqrt{3})^5} - \frac{1}{7(\sqrt{3})^7} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}\left(\frac{1}{\sqrt{3}}\right)^{2n+1} = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

(c) $\lim_{x\to 0}\frac{\tan(x)-\sin(x)}{x^2} = \lim_{x\to 0}\frac{\left(x+\frac{x^3}{3}-\cdots\right)-\left(x-\frac{x^3}{3!}+\cdots\right)}{x^2} = \lim_{x\to 0}\frac{\frac{x^3}{2}}{x^2} = 0$

(9.10 Ex8)

6. (8%) Let $f(\mathrm{x}) = x^6 e^{x^3}$. Try to evaluate the high order derivative $f^{(60)}(0)$

Ans:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^{x^3} = 1 + x^3 + \frac{x^6}{2!} + \cdots + \frac{x^{3n}}{n!} + \cdots$$

$$x^6 e^{x^3} = x^6 + x^9 + \frac{x^{12}}{2!} + \cdots + \frac{x^{3n+6}}{n!} + \cdots$$

$$f^{(60)}(0) = \frac{1}{18!} \times 60!$$

Theorem 10.11 (Slope in polar form)

If f is a differentiable function of θ, then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

provided that $\mathrm{d}x/\mathrm{d}\theta \neq 0$ at (r, θ). (See Figure 14.)

Theorem 10.13 (Area in polar coordinates)

If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,\mathrm{d}\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,\mathrm{d}\theta, \quad 0 < \beta - \alpha \leq 2\pi.$$

Theorem 10.14 (Arc length of a polar curve)

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Theorem 10.15 (Area of a surface of revolution)

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line as follows.

1. $S = 2\pi \int_\alpha^\beta y\, ds = 2\pi \int_\alpha^\beta f(\theta) \sin\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$
   About the polar axis
2. $S = 2\pi \int_\alpha^\beta x\, ds = 2\pi \int_\alpha^\beta f(\theta) \cos\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$
   About the line $\theta = \frac{\pi}{2}$

7. (8%) Find the area of the region which is inside the circle $r = 6\cos\theta$ and outside the cardioid $r = 2(1 + \cos\theta)$. (Both are represented in polar coordinates)

Ans:

Since $(r, -\theta)$ also lies on both graph, both of them are symmetric with respect to polar axis.

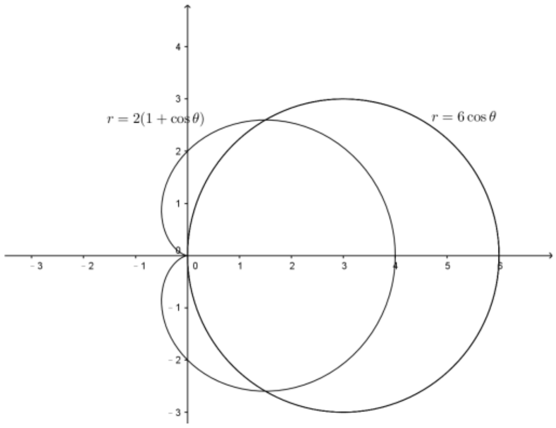
The intersection is $2(1+\cos\theta) = 6\cos\theta \rightarrow \cos\theta = \frac{1}{2}$, therefore, $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$.

The area can be calculated as $A = 2 \times \frac{1}{2}\left[\int_0^{\frac{\pi}{3}} (6\cos\theta)^2 d\theta - \int_0^{\frac{\pi}{3}} 4(1+\cos\theta)^2 d\theta\right] =$

$36\int_0^{\frac{\pi}{3}} \cos^2\theta d\theta - 4\int_0^{\frac{\pi}{3}} (1+\cos\theta)^2 d\theta = 36\int_0^{\frac{\pi}{3}} \frac{1+\cos(2\theta)}{2} d\theta - 4\int_0^{\frac{\pi}{3}} (1+\cos\theta)^2 d\theta =$

$\int_0^{\frac{\pi}{3}} (18\cos 2\theta + 18 - 4 - 8\cos\theta - 2\cos 2\theta - 2) d\theta = (9\sin 2\theta + 18\theta - 4\theta -$

$8\sin\theta - \sin 2\theta - 2\theta)\Big|_0^{\frac{\pi}{3}} = 4\pi$



8. (5%) Find the arc length of $r = e^{\theta}$ from $\theta = 0$ to $\theta = 2\pi$

Ans: $S = \int_0^{2\pi} \sqrt{(e^{\theta})^2 + (e^{\theta})^2}\, d\theta = \sqrt{2}\int_0^{2\pi} e^{\theta}\, d\theta = \sqrt{2}(e^{2\pi} - 1)$

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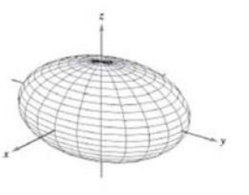
<td></td>
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Ellipsoid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Ellipse	Parallel to xz-plane
Ellipse	Parallel to yz-plane

The surface is a sphere if $a = b = c \neq 0$.

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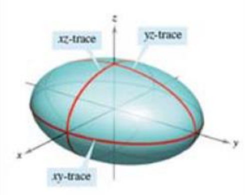
<td>
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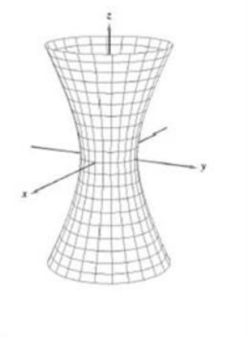
<td></td>
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Hyperboloid of One Sheet

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is negative.

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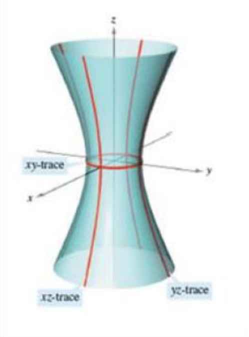
<td>
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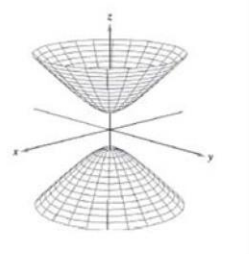
<td></td>
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Hyperboloid of Two Sheets

$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.

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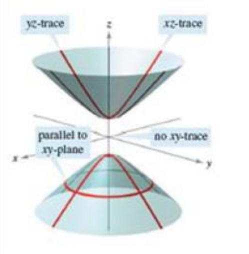
<td>
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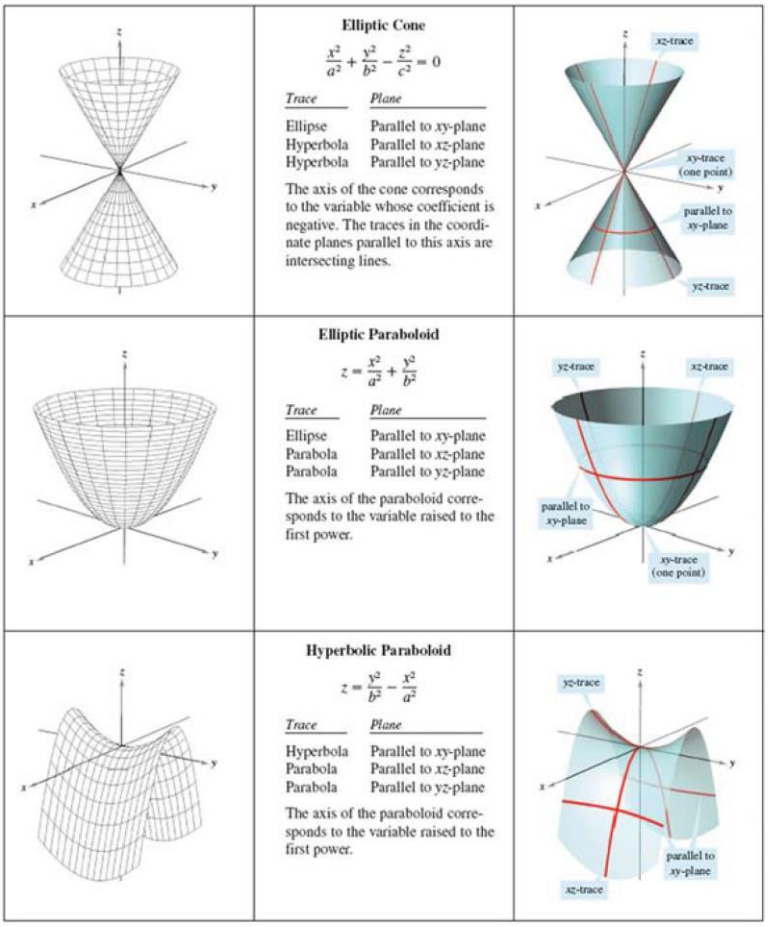


Elliptic Cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.

Elliptic Paraboloid

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.

Hyperbolic Paraboloid

$$z=\frac{y^2}{b^2}-\frac{x^2}{a^2}$$

Trace	*Plane*
Hyperbola	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.

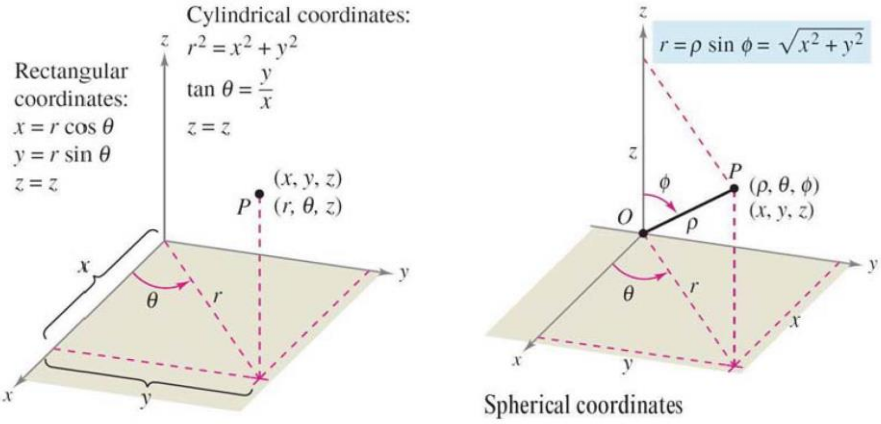
Definition 11.2 (Equation of cylinders)

The equation of a cylinder whose ruling are parallel to one of the coordinate axes contain only the variables corresponding to the other two axes.

Definition 11.4 (Surface of revolution)

If the graph of a radius function r is revolved about one of the coordinate axes, the equation of the resulting surface of revolution has one of the following forms.

1. Revolved about the x-axis: $y^2+z^2=[r(x)]^2$
2. Revolved about the y-axis: $x^2+z^2=[r(y)]^2$
3. Revolved about the z-axis: $x^2+y^2=[r(z)]^2$



9. (12%) Classify the following surface, if it is quadratic surface you should further classify it into six basic types of surface
   (a) $z = x^2 + 3y^2$
   (b) $x^2 + y^2 - 2z = 0$
   (c) $r^2 = z^2 + 2$ (this representation is in cylindrical coordinates)
   (d) $\rho = 4\sec(\Phi)$ (this representation is in spherical coordinates)

Ans:

(a) $z = x^2 + 3y^2 = x^2 + \frac{y^2}{(\frac{1}{\sqrt{3}})^2}$ which is an elliptic paraboloid

(b) $x^2 + y^2 - 2z = 0 \rightarrow x^2 + y^2 = 2z$ which is a surface of revolution or elliptic paraboloid

(c) $r^2 = z^2 + 2 \rightarrow x^2 + y^2 - z^2 = 2 \rightarrow \frac{x^2}{(\sqrt{2})^2} + \frac{y^2}{(\sqrt{2})^2} - \frac{z^2}{(\sqrt{2})^2} = 1$ which is a hyperboloid of one sheet

(d) $\rho = 4\sec(\Phi) \rightarrow z = \rho\cos(\Phi) = 4$ which is a plane

Definition 12.2 (The limit of a vector-valued function)

1. If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} \qquad \text{Plane}$$

provided f and g have limits as $t \to a$.

2. If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} + \left[\lim_{t\to a} h(t)\right]\mathbf{k} \qquad \text{Space}$$

provided f, g, and h have limits as $t \to a$.

Theorem 12.1 (Differentiation of vector-valued functions)

1. *If* $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$*, where* f *and* g *are differentiable functions of* t*, then*

$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j}. \qquad \textit{Plane}$$

2. *If* $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$*, where* f*,* g*, and* h *are differentiable functions of* t*, then*

$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}. \qquad \textit{Space}$$

Theorem 12.2 (Properties of the derivative)

Let $\mathbf{r}$ *and* $\mathbf{u}$ *be differentiable vector-valued functions of* t*, let* w *be a differentiable real-valued function of* t*, and let* c *be scalar.*

1. $D_t\,[c\,\mathbf{r}(t)] = c\,\mathbf{r}'(t)$
2. $D_t\,[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
3. $D_t\,[w(t)\,\mathbf{r}(t)] = w(t)\,\mathbf{r}'(t) + w'(t)\,\mathbf{r}(t)$
4. $D_t\,[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$
5. $D_t\,[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$
6. $D_t\,[\mathbf{r}\,(w(t))] = \mathbf{r}'\,(w(t))\,w'(t)$
7. *If* $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$*, then* $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

Definition 12.5 (Integration of vector-valued functions)

- If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, where f and g are continuous on $[a, b]$, then the indefinite integral(antiderivative) of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,\mathrm{d}t = \left[\int f(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int g(t)\,\mathrm{d}t\right]\mathbf{j} \qquad \text{Plane}$$

and its definite integral over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,\mathrm{d}t = \left[\int_a^b f(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int_a^b g(t)\,\mathrm{d}t\right]\mathbf{j}.$$

Definition 12.5 (continue)

- If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are continuous on $[a, b]$, then the indefinite integral (antiderivative) of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,\mathrm{d}t = \left[\int f(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int g(t)\,\mathrm{d}t\right]\mathbf{j} + \left[\int h(t)\,\mathrm{d}t\right]\mathbf{k} \qquad \text{Space}$$

and its definite integral over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,\mathrm{d}t = \left[\int_a^b f(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int_a^b g(t)\,\mathrm{d}t\right]\mathbf{j} + \left[\int_a^b h(t)\,\mathrm{d}t\right]\mathbf{k}.$$

10. (12%) Evaluate the following expression

(a) $\lim\limits_{t\to 1} \sqrt{t}\boldsymbol{i} + \frac{\ln t}{t^2-1}\boldsymbol{j} + \frac{1}{t-1}\boldsymbol{k}$

(b) $\lim\limits_{t\to 0} \frac{sin2t}{t}\boldsymbol{i} + e^{-t}\boldsymbol{j} + 5\boldsymbol{k}$

(c) Let $\boldsymbol{r}(\mathrm{t}) = 3\mathrm{t}\boldsymbol{i} + (\mathrm{t}-1)\boldsymbol{j}$, $\boldsymbol{u}(\mathrm{t}) = \mathrm{t}\boldsymbol{i} + t^2\boldsymbol{j} + \frac{2}{3}t^3\boldsymbol{k}$, find $\frac{d}{dt}[\boldsymbol{r}(t)\cdot\boldsymbol{u}(t)]$

(d) $\int(3\sqrt{t}\boldsymbol{i} + \frac{2}{t}\boldsymbol{j} + 6\boldsymbol{k})dt$

Ans:

(a) Does not exit, since $\lim_{t\to 1}\frac{1}{t-1}$ does not exist

(b) $\lim_{t\to 0}\frac{sin2t}{t}\boldsymbol{i}+e^{-t}\boldsymbol{j}+5\boldsymbol{k}=2\boldsymbol{i}+\boldsymbol{j}+5\boldsymbol{k}$

Since $\lim_{t\to 0}\frac{sin2t}{t}=\lim_{t\to 0}\frac{sin2t}{2t}2=2$

(c) $\boldsymbol{u}'(t)=\boldsymbol{i}+2\mathrm{t}\boldsymbol{j}+2\mathrm{t}^2\boldsymbol{k}$, $\boldsymbol{r}'(t)=3\boldsymbol{i}+\boldsymbol{j}$

$$\frac{d}{dt}[\boldsymbol{r}(t)\cdot\boldsymbol{u}(t)]=\boldsymbol{r}(t)\cdot\boldsymbol{u}'(t)+\boldsymbol{u}(t)\cdot\boldsymbol{r}'(t)$$

$$=[(3t)+(t-1)(2t)]+[(3t)+(t^2)]=4t+3t^2$$

(d) $\int(3\sqrt{t}\boldsymbol{i}+\frac{2}{t}\boldsymbol{j}+\boldsymbol{k})dt=2t^{\frac{3}{2}}\boldsymbol{i}+2\ln t\,\boldsymbol{j}+6t\boldsymbol{k}+\boldsymbol{C}$