Assignment 10

1. Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ using the appropriate Chain Rule.

$$w = x^2 + y^2 + z^2,\ x = t\sin s,\ y = t\cos s,\ z = st^2$$

2. Differentiate implicitly to find the first partial derivatives of a.

(a) $x\ln y + y^2 a + a^2 = 8$ (b) $a - \sqrt{x-y} - \sqrt{y-z} = 0$

3. Find the directional derivative of the function at P in the direction of $\mathbf{v}$.

$$f(x,y) = e^{-(x^2+y^2)},\ P(0,0),\ \mathbf{v} = \mathbf{i} + \mathbf{j}$$

4. Use the gradient to find the directional derivative of the function at P in the direction of $\overrightarrow{PQ}$.

$$f(x,y,z) = \ln(x+y+z),\ P(1,0,0),\ Q(4,3,1)$$

5. Find the gradient of the function and the maximum value of the directional derivative at the given point.

$$f(x,y) = \frac{x+y}{y+1},\ (0,1)$$

sol:

1.

$$\begin{aligned}\frac{\partial w}{\partial s} &= 2x + \cos s + 2y(-t\sin s) + 2z(t^2)\\ &= 2t^2\sin s\cos s - 2t^2\sin s\cos s + 2st^4\\ &= 2st^4\\ \frac{\partial w}{\partial t} &= 2x\sin s + 2y\cos s + 2z(2st)\\ &= 2t\sin^2 s + 2t\cos^2 s + 4s^2t^3\\ &= 2t + 4s^2t^3\end{aligned}$$

2. (a)

$$\begin{aligned}\frac{\partial a}{\partial x} &= \frac{-F_x(x,y,a)}{F_a(x,y,a)} = \frac{-\ln y}{y^2+2a}\\ \frac{\partial a}{\partial y} &= \frac{-F_y(x,y,a)}{F_a(x,y,a)} = -\frac{(x/y)+2ya}{y^2+2a} = -\frac{x+2y^2a}{y^3+2ya}\end{aligned}$$

(b)

$$\begin{aligned}\frac{\partial a}{\partial x} &= \frac{-F_x}{F_a} = \frac{1}{2}\frac{(x-y)^{-1/2}}{1} = \frac{1}{2\sqrt{x-y}}\\ \frac{\partial a}{\partial y} &= \frac{-F_y}{F_a} = \frac{-1}{2}(x-y)^{-1/2} + \frac{1}{2}(y-z)^{-1/2} = \frac{-1}{2\sqrt{x-y}} + \frac{1}{2\sqrt{y-z}}\\ \frac{\partial a}{\partial z} &= \frac{-F_z}{F_a} = \frac{-1}{2\sqrt{y-z}}\end{aligned}$$

3.

$$\begin{aligned}\mathbf{u} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} \\ D_u f(x,y) &= -2xe^{-(x^2+y^2)}\left(\frac{\sqrt{2}}{2}\right) + \left(-2ye^{-(x^2+y^2)}\right)\left(\frac{\sqrt{2}}{2}\right) \\ D_u f(0,0) &= 0\end{aligned}$$

4.

$$\begin{aligned}\mathbf{v} &= 3\mathbf{i} + 3\mathbf{j} + \mathbf{k} \\ \nabla f &= \frac{1}{x+y+z}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ \text{At } (1,0,0)\ , \nabla f &= \mathbf{i} + \mathbf{j} + \mathbf{k} \\ \mathbf{u} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{19}}(3\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \\ D_u f &= \nabla f \cdot \mathbf{u} = \frac{7}{\sqrt{19}} = \frac{7\sqrt{19}}{19}\end{aligned}$$

5.

$$\begin{aligned}\nabla f(x,y) &= \frac{1}{y+1}\mathbf{i} + \frac{1-x}{(y+1)^2}\mathbf{j} \\ \nabla f(0,1) &= \frac{1}{2}\mathbf{i} + \frac{1}{4}\mathbf{j} \\ \|\nabla f(0,1)\| &= \sqrt{\frac{1}{4} + \frac{1}{16}} = \frac{1}{4}\sqrt{5}\end{aligned}$$