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Chapter 3

Applications of Differentiation

3.1 Summary

1. **Extrema** Let f be defined on an interval I containing c.

   1. $f(c)$ is the **minimum** (最小值) of f on I if $f(c) \leq f(x)$ for all x in I.

2. $f(c)$ is the **maximum** (最大值) of f on I if $f(c) \geq f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the **extreme values** (極值), or **extrema** (極值) (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** (絕對最小值) and **absolute maximum** (絕對最大值), or the **global minimum** (全域最小值) and **global maximum** (全域最大值), on the interval. 5

2. **The Extreme Value Theorem** (極值定理)] If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

A theorem which guarantees the existence of an absolute max and an absolute min for any continuous function over a closed interval. 6

3. **Relative extrema** (**相對極值**)

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** (**相對極大值**) of f, or you can say that f has a relative maximum at $(c, f(c))$.
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** (**相對極小值**) of f, or you can say that f has a relative minimum at $(c, f(c))$.

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima. Relative maximum and relative minimum are sometimes called **local maximum** (**局部極大值**) and **local minimum** (**局部極小值**), respectively. 9

4. **Critical number** Let f be defined at c. If $f'(c) = 0$ or if f is

not differentiable at c, then c is a **critical number** (**臨界數**) of f. . 13

5. **Relative extrema occur only at critical numbers** If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f. 14

6. **Guidelines for finding extrema on a closed interval** To find the extrema of a continuous function f on a closed interval $[a, b]$, use the following steps.

   (a) Find the critical numbers of f in (a, b).

   (b) Evaluate f at each critical number in (a, b).

   (c) Evaluate f at each endpoint of $[a, b]$.

   (d) The least of these values is the minimum. The greatest is the maximum.

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Section 3.2 Rolle's Theorem and the Mean Value Theorem 24

7. **Rolle's Theorem (洛爾定理)** Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If

$$f(a) = f(b)$$

then there is at least one number c in (a,b) such that $f'(c) = 0$.
A theorem of calculus that ensures the existence of a critical point between any two points on a "nice" function that have the same y-value.
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8. **Mean Value Theorem (均值定理)** If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then

there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

A major theorem of calculus that relates values of a function to a value of its derivative. Essentially the theorem states that for a "nice" function, there is a tangent line parallel to any secant line. 35

Section 3.3 Increasing and decreasing functions and the First Derivative Test 45

9. **Increasing and decreasing functions**

A function f is **increasing** (遞增) on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
A function f is **decreasing** (遞減) on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$. 46

10. **Test for increasing and decreasing functions (函數遞增、遞減的微分**

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

(a) If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.

(b) If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.

(c) If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

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11. **Guidelines for finding intervals on which a function is increasing or**

Let f be continuous on the interval (a, b). To find the open intervals on which f is increasing or decreasing, use the following steps.

(a) Locate the critical numbers of f in (a, b), and use these numbers to determine test intervals.

(b) Determine the sign of $f'(x)$ at one test value in each of the intervals.

(c) Use Theorem 3.5 to determine whether f is increasing or decreasing on each interval.

These guidelines are also valid if the interval (a, b) is replaced by an interval of the form $(-\infty, b)$, (a, ∞), or $(-\infty, \infty)$. 52

12. **The First Derivative Test (一階導數檢定)** Let c a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.

(a) If $f'(x)$ changes from negative to positive at c, then f has a **relative minimum (相對極小值)** at $(c, f(c))$.

(b) If $f'(x)$ changes from positive to negative at c, then f has a

relative maximum (**相對極大值**) at $(c, f(c))$.

(c) If $f'(x)$ is positive on both sides of c or negative on both sides of c, then f is neither a relative minimum nor a relative maximum.

13. **Concavity** Let f be differentiable on an open interval I. The graph of f is **concave upward** (**凹向上**) on I if f' is increasing on the interval and **concave downward** (**凹向下**) on I if f' is decreasing on the interval. 72

14. **Test for concavity (凹性檢驗)** Let f be a function whose second derivative exists on an open interval I.

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

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15. **Point of inflection** Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a **point of inflection** (反曲點) of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point. 83

16. **Points of inflection** If $(c, f(c))$ is a point of inflection of the

graph of f, then either $f''(c) = 0$ or f'' does not exist at $x = c$. 84

17. **Second Derivative Test (第二階導數檢定)** that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

(a) If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.

(b) If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.

If $f''(c) = 0$, the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

A method for determining whether a critical point is a relative minimum or maximum. 86

Section 3.5 Limits at infinity 90

18. **Limits at infinity** Let L be a real number.

(a) The statement $\lim_{x\to\infty} f(x) = L$ means that for each $\epsilon > 0$ there exists an $M > 0$ such that $|f(x) - L| < \epsilon$ whenever $x > M$.

(b) The statement $\lim_{x\to-\infty} f(x) = L$ means that for each $\epsilon > 0$ there exists an $N < 0$ such that $|f(x) - L| < \epsilon$ whenever $x < N$.

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19. **Horizontal asymptote** : The line $y = L$ is a **horizontal asymptote** (**水平漸近線**) of the graph of f if

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L.$$

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20. **Limits at infinity** If r is a positive rational number and c is any

real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0.$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0.$$

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21. Guidelines for finding limits at $\pm\infty$ of rational functions

(a) If the degree of the numerator is less than the degree of the denominator, then the limit of the rational function is 0.

(b) If the degree of the numerator is equal to the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.

(c) If the degree of the numerator is greater than the degree of the denominator, then the limit of the rational function does not exist.

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22. **Limits at $\pm\infty$ of rational functions** Let $f(x) = \dfrac{p(x)}{q(x)}$ be a **rational function** (有理函數) where $p(x) = a_m x^m + \cdots + a_0$ and $q(x) = b_n x^n + \cdots + b_0$ are polynomials. Then

$$\lim_{x\to\infty} r(x) = \begin{cases} 0 & \text{if } m < n \\ \dfrac{a_m}{b_m} & \text{if } m = n \\ \operatorname{sgn}\left(\dfrac{a_m}{b_n}\right)\infty & \text{if } m > n \end{cases}$$

$$\lim_{x \to -\infty} r(x) = \begin{cases} 0 & \text{if } m < n \\ \dfrac{a_m}{b_m} & \text{if } m = n \\ (-1)^{m-n} \operatorname{sgn}\left(\dfrac{a_m}{b_n}\right) \infty & \text{if } m > n \end{cases}$$

23. **Infinite limits at infinity** Let f be a function defined on the interval (a, ∞).

(a) The statement $\lim_{x \to \infty} f(x) = \infty$ means that for each positive number M, there is a corresponding number $N > 0$ such that $f(x) > M$ whenever $x > N$.

(b) The statement $\lim_{x \to \infty} f(x) = -\infty$ means that for each negative number M, there is a corresponding number $N > 0$ such that $f(x) < M$ whenever $x > N$.

24. **Guidelines for analyzing the graph of a function**

(a) Determine the domain and range of the function.

(b) Determine the intercepts, asymptotes, and symmetry of the graph.

(c) Locate the x-values for which $f'(x)$ and $f''(x)$ either are zero or do not exist. Use the results to determine relative extrema and points of inflection.

25. **Slant asymptote** The line $y = mx + b$ is a **slant asymptote** (**歪斜漸近線**) of the graph of f if

$$\lim_{x\to-\infty} f(x) - (mx+b) = 0 \quad \text{or} \quad \lim_{x\to\infty} f(x) - (mx+b) = 0.$$

26. **Slant asymptote** : If the line $y = mx + b$ is a **slant asymptote** (**歪斜漸近線**) of the graph of f, then

$$m = \lim_{x\to-\infty} \frac{f(x)}{x} \qquad b = \lim_{x\to-\infty} f(x) - mx$$

or

$$m = \lim_{x\to\infty} \frac{f(x)}{x} \qquad b = \lim_{x\to\infty} f(x) - mx.$$

27. **Guidelines for solving applied minimum and maximum problems**

(a) Identify all given quantities and all quantities to be determined. If possible, make a sketch.

(b) Write a **primary equation** (主要方程式) for the quantity that is to be maximized or minimized. (A review of several useful formulas from geometry is presented inside the back cover.)

(c) Reduce the primary equation to one having a single independent variable. This may involve the use of **secondary equations** (次要方程式) relating the independent variables of the primary equation.

(d) Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.

(e) Determine the desired maximum or minimum value by the calculus techniques discussed in Sections ?? through ??.

28. **Newton's method for approximating the zeros of a function** (牛頓法

Let $f(c) = 0$, where f is differentiable on an open interval containing c. Then, to approximate c, use the following steps.

(a) Make an initial estimate x_1 that is close to c. (A graph is helpful.)

(b) Determine a new approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(c) If $|x_n - x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedure is called an **iteration** (迭代). 169

29. **Fixed-point Convergence Theorem (固定點收斂定理)** Suppose that the function g has a **fixed point** (固定點) c, i.e. $c = g(c)$ and that there is a number α such that

(i) g is continuous in $[c - \alpha, c + \alpha]$,

31. **Differentials** Let $y = f(x)$ represent a function that is differentiable on an open interval containing x. The **differential** (微分) of x (denoted by $\mathrm{d}x$) is any nonzero real number. The differential of y (denoted by $\mathrm{d}y$) is

$$\mathrm{d}y = f'(x)\,\mathrm{d}x.$$

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32. **Differential formulas (微分公式)** Let u and v be differentiable functions of x.

Constant multiple: $\mathrm{d}[cu] = c\,\mathrm{d}u$

Sum or difference: $\mathrm{d}[u \pm v] = \mathrm{d}u \pm \mathrm{d}v$

Product: $\mathrm{d}[uv] = u\,\mathrm{d}v + v\,\mathrm{d}u$

Quotient: $\mathrm{d}\left[\frac{u}{v}\right] = \frac{v\,\mathrm{d}u - u\,\mathrm{d}v}{v^2}$

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