Calculus

Chien-Hong Cho 卓建宏

Department of Applied Mathematics
National Sun Yat-sen University

September 15, 2021

Notations

1. $A, B, \cdots$: sets; $a, b, \cdots$: elements.

Definition

A set is a collection of objects.

Remark: A set is often represented in the following ways:

1. A is the set of all integers. (State directly).
2. $A = \{\text{objects} \,|\, \text{conditions}\} = \{\text{objects} : \text{conditions}\}$.
3. $A = \{\text{List all the members}\}$.

2. "belong to" is denoted by "$\in$".
3. "not belong to" is denoted by "$\notin$". Ex: $a \in A$: a belongs to A; $b \notin B$: b is not in B.

4. $A \subseteq B$: Set A is contained in set B; $A \supseteq B$: A contain B.

   $\not\subseteq$: not contain; $\subsetneq$: properly contain.
5. "because" is denoted by "$\because$".
6. "therefore" is denoted by "$\therefore$".
7. . "for all" is denoted by "$\forall$".
8. "exists" or "there is at least one" is denoted by "$\exists$".

   "there exists one and only one (unique)": $\exists!$ (or $\exists 1$).
9. "such that" is denoted by "s.t.".
10. α: alpha; β: beta; γ: gamma; δ: delta; λ: lambda; θ: theta; φ: phi;

    ψ: psi; σ: sigma; ω: omega; ε: epsilon; Γ: Gamma; Λ: Lambda;

    Σ: Sigma; Ω: Omega.

Number Systems

1. $\mathbb{N} = \{1, 2, \cdots\}$: The positive integers (natural number).
2. $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$: The integers.
3. $\mathbb{Q} = \{\frac{q}{p} | p, q \in \mathbb{Z} \,\&\, p \neq 0\}$: The rational numbers.
4. $\mathbb{R} = \mathbb{Q} \cup \{\text{irrational numbers}\}$: The real numbers.

$$\underline{\mathbb{N} \subsetneqq \mathbb{Z} \subsetneqq \mathbb{Q} \subsetneqq \mathbb{R}}.$$

Definition

Let $-\infty < a < b < \infty$ be real numbers.

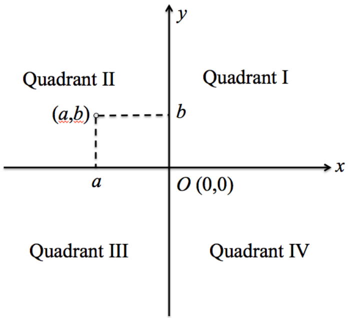
1. $(a, b) = \{x \in \mathbb{R} | a < x < b\}$: The open interval with end points a, b.
2. $(a, b] = \{x \in \mathbb{R} \mid a < x \leqslant b\}$: The half-open interval with end points a, b.
3. $[a, b) = \{x \in \mathbb{R} \mid a \leqslant x < b\}$: The semi-open interval with end points a, b.
4. $[a, b] = \{x \in \mathbb{R} | a \leqslant x \leqslant b\}$: The closed interval with end points a, b.
5. $(a, \infty) = \{x \in \mathbb{R} | a < x\}$.
6. $[a, \infty) = \{x \in \mathbb{R} | a \leqslant x\}$.
7. $(-\infty, a) = \{x \in \mathbb{R} | x < a\}$.
8. $(-\infty, a] = \{x \in \mathbb{R} | x \leqslant a\}$.
9. $\mathbb{R} = (-\infty, \infty)$.

Sec. 0.1: Graphs and Models

Definition (The Cartesian Plane (The rectangular coordinate system))

A plane used to graphically represent order pairs of real numbers, formed by two real number lines intersecting at right angles.

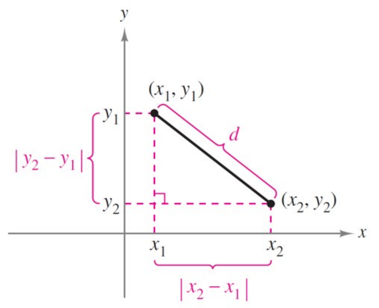
O: origin; (a, b): point, ordered pair;
The horizontal real line: x-axis;
The vertical real line: y-axis;
a: abscissa, x-coordinate;
b: ordinate, y-coordinate.
$(+, +)$: quadrant I; $(-, +)$: quadrant II;
$(-, -)$: quadrant III; $(+, -)$: quadrant IV;



Definition (The Distance Formula:)

- The distance between the points (x_1, y_1) and (x_2, y_2) in the plane is

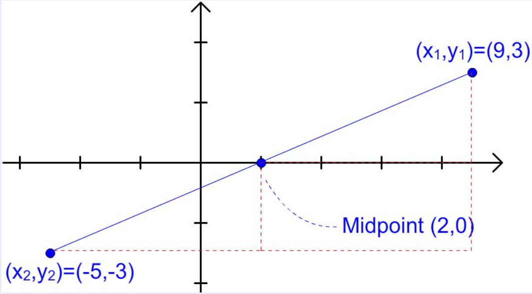
$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



Definition (The Midpoint Formula:)

- The midpoint of the segment jointing the points (x_1, y_1) and (x_2, y_2) is

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



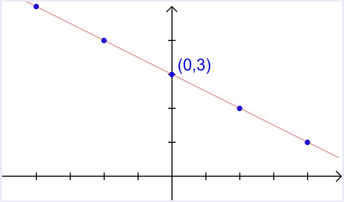
Graphs of Equations

Example (Graph of Equations)

Sketch the graph of the equation $y = 3 - 0.5x$.

Solution

x	-4	-2	0	2	4
y	5	4	3	2	1



Key Points

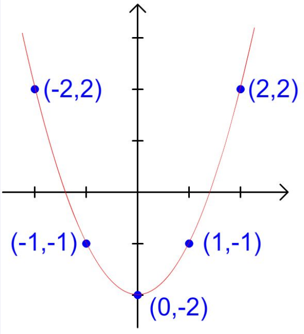
- Construct a table of values
- Plot these points
- Connect the points

Exercise

Sketch the graph of $y = x^2 - 2$

Solution

x	-2	-1	0	1	2
y	2	-1	-2	-1	2



Intercepts of a Graph

Definition

1. The x-intercept of a line L is a (or $(a, 0)$) if L intersects x-axis at $(a, 0)$.
2. The y-intercept of a line L is b (or $(0, b)$) if L intersects y-axis at $(0, b)$.

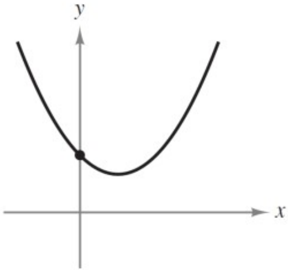


Figure: One y-intercept, No x-intercept

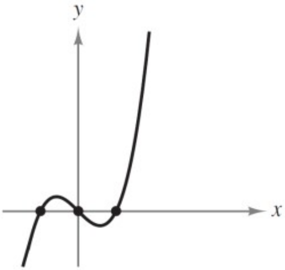


Figure: Three x-intercepts, One y-intercept

Intercepts of a Graph

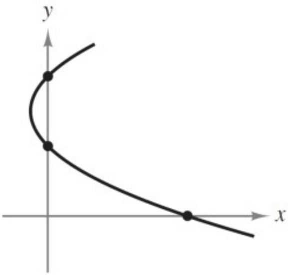


Figure: One x-intercept, Two y-intercepts

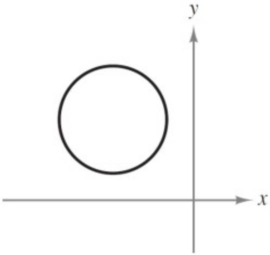


Figure: No intercepts

Finding intercepts

- To find x-intercepts, let $y = 0$ and solve the equation for x.
- To find y-intercepts, let $x = 0$ and solve the equation for y.

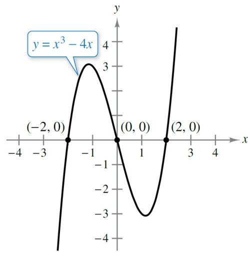
Example

Find the x- and y-intercepts of the graph of $y = x^3 - 4x$.

Solution

- *Let* $y = 0$. *Then*
$$0 = x^3 - 4x = x(x+2)(x-2),$$
$$\Longrightarrow x = 0, 2, -2$$
$\Longrightarrow$ *Three x-intercepts :(0,0), (-2,0), and (2,0)*

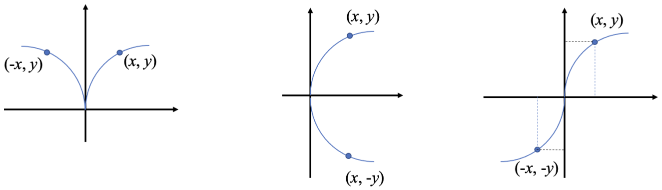
- *Let* $x = 0$. $\Longrightarrow$ *One y-intercept:* $(0, 0)$.



Symmetry of a Graph

Definition

1. A graph is symmetric to y-axis if $(-x, y)$ is on the graph whenever (x, y) is on the graph.
2. A graph is symmetric to x-axis if $(x, -y)$ is on the graph whenever (x, y) is on the graph.
3. A graph is symmetric to the origin if $(-x, -y)$ is on the graph whenever (x, y) is on the graph.



Example

Example

Test the graph $y = x^3 - 3x$ for symmetry with respect to (a) the y-axis; (b) the origin.

Solution

(a) *Test whether the point $(-x, y)$ is on the graph.*
$y = (-x)^3 - 3(-x) = -x^3 + 3x$*: is not an equivalent equation.*

(a) *Test whether the point $(-x, -y)$ is on the graph.*
$-y = (-x)^3 - 3(-x)$ *which implies* $y = x^3 - 3x$*: is an equivalent equation.*

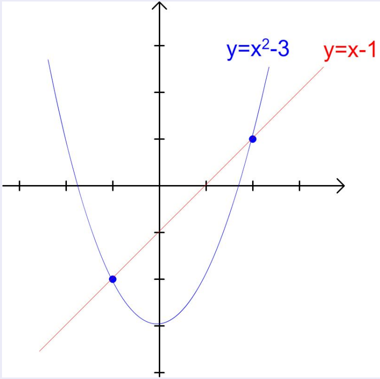
Points of Intersection

Example (1)

Show the graphs of $y = x^2 - 3$ and $y = x - 1$ have two points of intersections $(2,1)$ and $(-1,-2)$

Solution

$$\begin{aligned} & y = x^2 - 3 = x - 1 \\ \Rightarrow\ & x^2 - x - 2 = 0 \\ \Rightarrow\ & (x-2)(x+1) = 0 \\ \Rightarrow\ & x = 2 \ or\ -1 \\ \Rightarrow\ & (x,y) = (2,1)\ or\ (-1,-2) \end{aligned}$$



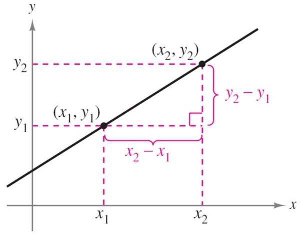
The slope of a line

Definition

The “slope” of the line passing through the points (x_1, y_1) and (x_2, y_2) is

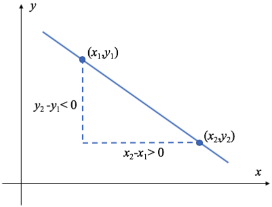
$$m = \frac{y_2 - y_1}{x_2 - x_1} \equiv \frac{\Delta y}{\Delta x}$$

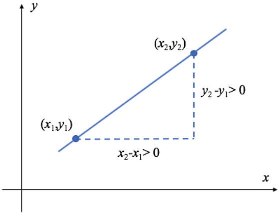
where $x_1 \neq x_2$. Here, $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ denote the horizontal and vertical changes respectively.

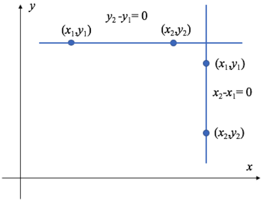


Remarks

- If the line falls from left to right, then $m < 0$.
- If the line rises from left to right, then $m > 0$.
- If the line is horizontal, then $m = 0$.
- If the line is vertical, then m is undefined.







Examples

1. Find the slope of the line which passes through $(-1, 1)$ and $(5, 3)$.

   Clearly, $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3 - 1}{5 - (-1)} = \dfrac{1}{3}$.

2. Find the slope of the line which passes through $(-2, 5)$ and$(3, 5)$.

   Clearly, $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{5 - 5}{3 - (-2)} = 0$.

3. Assume that $(1, -3), (3, 5), (5, b), (a, 1)$ lie on the same line. Find a, b.

   $\because \dfrac{b - 5}{5 - 3} = \dfrac{5 - (-3)}{3 - 1} = 4 \ \Rightarrow \ b = 13.$ (linear extrapolation)

   $\because \dfrac{1 - (-3)}{a - 1} = \dfrac{5 - (-3)}{3 - 1} = 4 \ \Rightarrow \ a = 2.$ (linear interpolation)

Remark

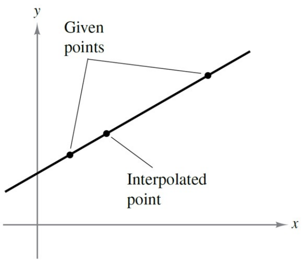


Figure: 內分點

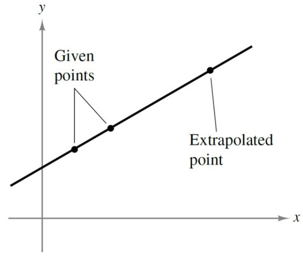


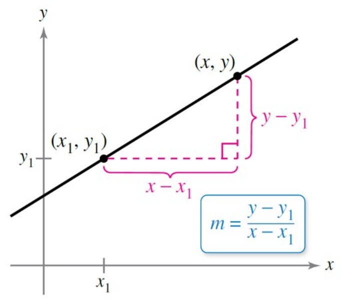
Figure: 外分點

Writing Linear Equation.

Point-Slope form (點斜式)

The “point-slope” form of the equation with slope m, passing through the point (x_1, y_1) is

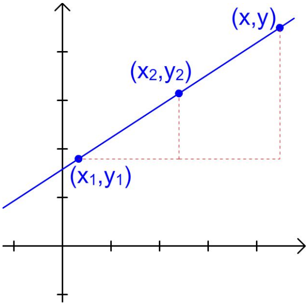
$$y - y_1 = m(x - x_1) \qquad (\because \frac{y - y_1}{x - x_1} = m, \text{ if } x \neq x_1)$$



Two-Point form (兩點式)

The "two-point" form of the equation of a line passing through points (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$ is

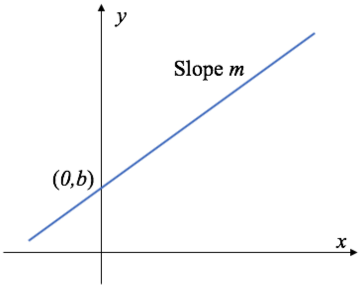
$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$



Slope-Intercept Form (斜截式)

The "slope-intercept" form of the equation of a line whose slope is m and whose y-intercept is $(0, b)$ is $y = mx + b$.

Remark: $\because y - b = m(x - 0)$.



Example

Find equation of the line that passes through $(1,3)$ and has slope $m=2$.

sol.$\because 2 = m = \frac{y-3}{x-1}$. $\therefore y-3 = 2(x-1) \Rightarrow y = 2(x-1)+3$.

Example

Find an equation of the line passing through $(1,5)$ and $(-3,7)$, and its corresponding slope-intercept form.

sol.

- $y-5 = \left(\frac{7-5}{-3-1}\right)(x-1) = -\frac{1}{2}(x-1)$.
- The slope-intercept form is $y = -\frac{1}{2}x + \frac{11}{2}$.

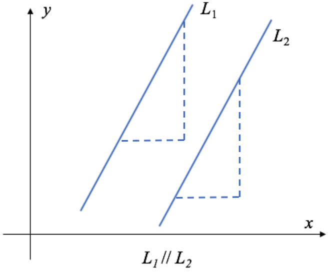
Example

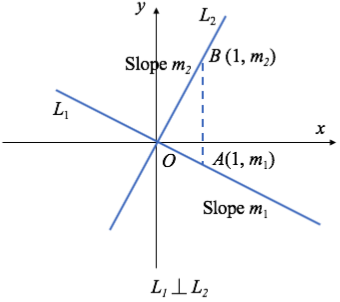
Find an equation of the line that has slope 3 and y-intercept -4.

sol. $y = 3x + (-4)$.

Remarks

- The general form of a line is $ax + by + c = 0$ and its slope is $-a/b$ if $b \neq 0$. ($\because y = -\frac{a}{b}x - \frac{c}{b}$).
- A equation of the form $ax + by + c = 0$ is called a linear equation since the graph is a straight line.
- Vertical line: $x = a$.
- Horizontal line: $y = b$.
- Let the slopes of lines L_1, L_2 be m_1, m_2, respectively.
  - (a) If L_1 and L_2 are parallel, then $m_1 = m_2$.
  - (b) If L_1 and L_2 are perpendicular (垂直), then $m_1 \cdot m_2 = -1$.





- A parallel shift of the line does not change its slope (平移不影響斜率)
- $\overline{OA}^2 + \overline{OB}^2 = \overline{AB}^2$
  $\Rightarrow \left(\sqrt{(1-0)^2+(m_1-0)^2}\right)^2 + \left(\sqrt{(1-0)^2+(m_2-0)^2}\right)^2 = (m_1-m_2)^2$
  $\Rightarrow m_1 \cdot m_2 = -1.$

Parallel and Perpendicular

Let L be the line: $ax + by = c$. ($m = -\frac{a}{b}$)

(a) The equation of a line which is parallel to L can be assumed as $ax + by = d$.

(b) The equation of a line which is perpendicular to L can be assumed as $bx - ay = d$. ($m = \frac{b}{a}$)

Example

Find equations of the lines that pass through the point $(2, -1)$ and is (a) parallel to and (b) perpendicular to the line $3x - y = -1$.

sol. (a) Let the equation be $3x - y = d$. $\therefore d = 3 \cdot 2 - (-1) = 7$.

(b) Let the equation be $x + 3y = d$. $\therefore d = 2 + 3 \cdot (-1) = -1$.

Functions

Definition (Functions)

$$\begin{array}{c}\text{Input}\\ x \in X\end{array} \Longrightarrow \boxed{\text{Function}} \Longrightarrow \begin{array}{c}\text{Output}\\ y \in Y\end{array}$$

- A function can be though of as a machine that inputs values of the independent variable and outputs values of the dependent variable.

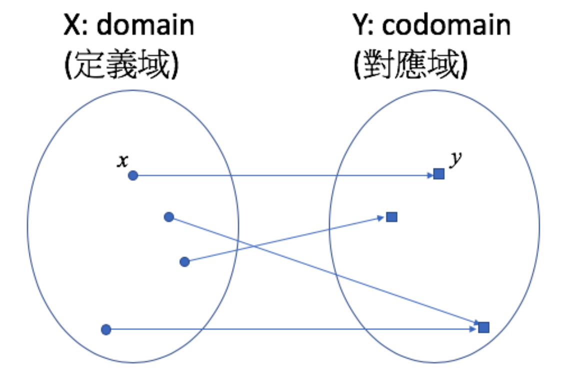
Example

$$y = 2 + 8x - 3x^2$$

- independent variable (自變數): x
- dependent variable (應變數): y

Definition

A function is a correspondence between a first set X, called the domain, and a second set Y, called the codomain, such that each member of the domain corresponds to exactly one member in the codomain.



Definition

The name of the function

$$\overset{\displaystyle\downarrow}{f} : X \to Y, \quad y = \underset{\displaystyle\uparrow}{f(x)}$$

function notation

Example

$$y = \frac{1-x}{2} \implies \text{we write } f(x) = \frac{1-x}{2}$$

- As $x = 3, f(3) = \frac{1-3}{2} = -1$.
- The value $f(x)$ is called a function value.

Example

Let $f(x) = x^2 - 2x$. Find the value of the function at $x = 2$ and evaluate the expressions $f(x+\Delta x)$ and $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$.

sol. $f(2) = 2^2 - 2 \cdot 2 = 0$.

$$f(x+\Delta x) = (x+\Delta x)^2 - 2(x+\Delta x) = x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x.$$

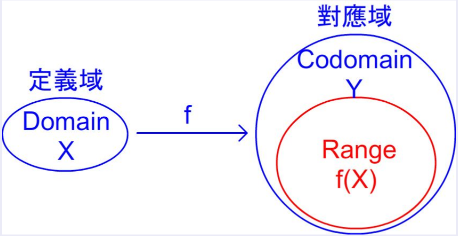
$$\begin{aligned}\frac{f(x+\Delta x) - f(x)}{\Delta x} &= \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x - (x^2 - 2x)}{\Delta x} \\ &= \frac{2x\Delta x + (\Delta x)^2 - 2\Delta x}{\Delta x} \\ &= 2x + \Delta x - 2.\end{aligned}$$

Definition

Let X, Y be two nonempty sets.

f is function from X to Y ($f : X \longmapsto Y$) if f assigns exactly one element y in Y to each element x in X. In this case, we write $y = f(x)$. (read as "f of x")

- $D(f) = X$: domain of f – The set of all values of the independent variables.
- Y: codomain of f.
- $R(f) = f(X) \equiv \{y \in Y \mid \exists x \in X \text{ s.t. } y = f(x)\} \subseteq Y$: range (image) of f – The set of all values taken on the independent variables.



Example

Find the domain and range of each function.

a. $y = f(x) = \sqrt{x-1}$

b. $y = f(x) = \begin{cases} 1-x, & x < 1 \\ \sqrt{x-1}, & x \geq 1 \end{cases}$

sol.

a. Need $x - 1 \geq 0$! $\therefore$ The domain is $[1, \infty)$.
   The range is $[0, \infty)$.

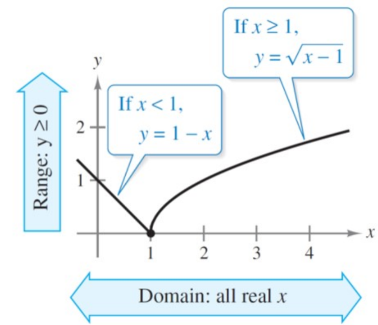
b. The domain is $\mathbb{R}$.
   The range is $[0, \infty)$.

Remark

Unless explicitly stated otherwise, the domain of a function f is the largest set in $\mathbb{R}$ for which f is defined.

Remark

The function $f(x) = \begin{cases} 1 - x, & x < 1 \\ \sqrt{x-1}, & x \geq 1 \end{cases}$
is a piecewise-defined function.



Definition

A piecewise-defined function is a function that is defined by two or more equations, each over a specified domain.

Example

Decide whether y is a function of x:

$(a)\ x^2 + y = 1$; $(b)\ x + y^2 = 1$; $(c)\ x^2 + y^2 = 1$; $(d)\ x^2y + xy = 1$;

sol. $\underline{\forall x \in \text{ domain }, \ \exists!\ y \text{ s.t. } y = f(x)}$

(a) $\because y = 1 - x^2$. $\quad \therefore y$ is a function of x.

(b) $\because y = \pm\sqrt{1 - x}$, not unique!! $\quad \therefore y$ is not a function of x.

(c) $\because y = \pm\sqrt{1 - x^2}$, not unique!! $\quad \therefore y$ is not a function of x.

(d) $\because y = \dfrac{1}{x^2 + x}$. $\quad \therefore y$ is a function of x. (if $x \neq 0$!!)

i.e.: $D(f) = \mathbb{R} - \{0\}$ if we put $y = f(x)$.

Remark

Question

When will y be a function of x?

Answer

$$\forall \text{ input } x \ (\forall\, x \in \text{ domain }), \ \ \exists! \text{ output } y.$$

Question

How to decide whether a curve defines a function of x?

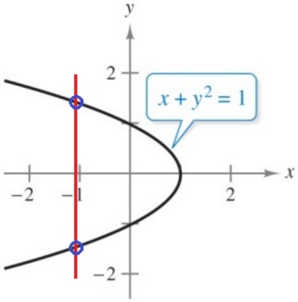
Answer

The vertical line test: Every vertical line intersects the graph at most once.

Example

Decide whether $x + y^2 = 1$ represents y as a function of x.

sol. No, some values of x determine two values of y.



Example

Decide whether each graph represents y as a function of x.

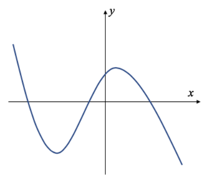


Figure: (a)

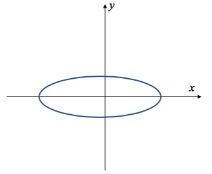


Figure: (b)

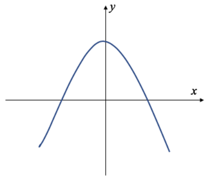


Figure: (c)

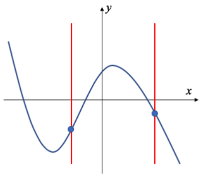


Figure: (a) Yes!

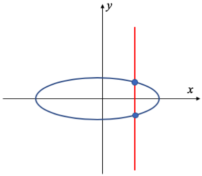


Figure: (b) No!

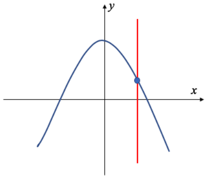
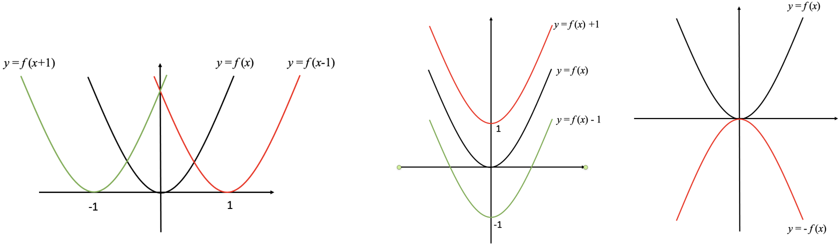


Figure: (c) Yes!

Transformations of Functions

Let $f(x) = x^2$.



Horizontal shift Vertical shift Reflection

Transformations of Functions

Basic Types of Transformations

Let $y = f(x)$.

- Horizontal shift c units to the right: $y = f(x - c)$.
- Vertical shift c units downward: $y = f(x) - c$.
- Reflection about $x-$ axis: $y = -f(x)$.
- Reflection about $y-$ axis: $y = f(-x)$.
- Reflection about the origin: $y = -f(-x)$.

Definition ($1-1$ (one-to-one))

Let $f : X \longmapsto Y$ be a function. f is said to be one-to-one ($1-1$) if for all $y \in f(X)$, there exists exactly one $x \in X$ such that $y = f(x)$.

Or equivalently, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Or equivalently, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Example

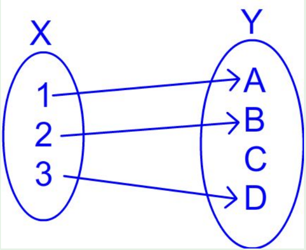


Figure: 1-1

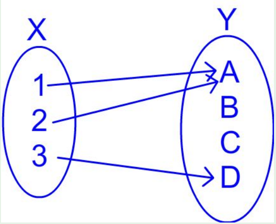


Figure: NOT 1-1

Remark

A function f is not $1-1$ if

$$\exists x_1 \neq x_2 \in D(f) \text{ s.t. } f(x_1) = f(x_2).$$

Example

Decide whether the following functions are $1-1$:

(a) $f(x) = 2x - 3$; (b) $f(x) = x^2 + 1$.

sol. (a) We need to check: if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Let $f(x_1) = f(x_2)$. $\therefore 2x_1 - 3 = 2x_2 - 3 \Rightarrow x_1 = x_2$.

$\therefore f$ is $1-1$.

(b) $\because f(-1) = (-1)^2 + 1 = 2$ & $f(1) = 1^2 + 1 = 2$ but $1 \neq -1$.

$\therefore f$ is not $1-1$.

Horizon Line Test:

A function is $1-1$ if every horizontal line intersects the graph of the function at most once.

Example

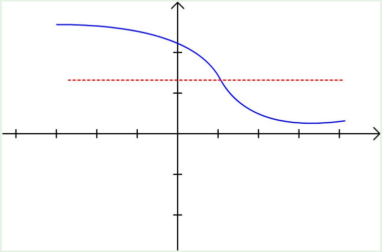


Figure: 1-1

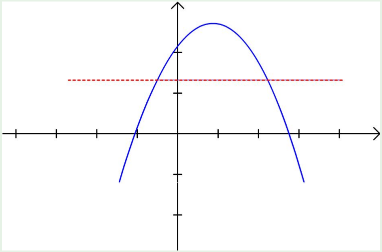


Figure: NOT 1-1

Combinations of Functions

- Two functions can be combined by the operations of $+, -, \times, \div$ to create new functions. For instance, $f(x) \pm g(x)$, $f(x)g(x)$, $f(x)/g(x)$ and etc..

Example

1. $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ $(a_n \neq 0)$ is a polynomial function.

   n is a nonnegative integer and is called the degree of the polynomial function f, which is denoted by $n = \deg f(x)$.

2. Let $p(x),\ q(x)$ be polynomials functions with $q(x) \neq 0$. Then $f(x) = \dfrac{p(x)}{q(x)}$ is called a rational function.

Definition

$f(x)$ is called an algebraic function if it can be expressed as a finite number of sums, differences, multiples, quotients and radicals involving x^n $(\text{ex}: \sqrt{x}, \cdots)$. Otherwise, it is called transcendental. $(\text{ex}: \sin x, \cos x, \cdots)$

Combinations of Functions

- $$\text{Input } x \in X \Longrightarrow \boxed{\text{Function } g} \Longrightarrow \text{Output } g(x) \Longrightarrow \boxed{\text{Function } f} \Longrightarrow \text{Output } f(g(x))$$

Example

Let $f(x) = x - 1$ and $g(x) = x^2$. Then

$h(x) = f(x) - g(x) = -x^2 + x - 1$and $k(x) = f(g(x)) = g(x) - 1 = x^2 - 1$.

Definition

Let $g : X \to Y$ $(y = g(x))$ and $f : W \to Z$ $(z = f(w))$ be two functions such that $R(g) \subseteq D(f)$. Then, the composite function of f and g is defined by

$$f \circ g : X \to Z \quad \text{s.t.} \quad z = (f \circ g)(x) = f(g(x)).$$

Remarks

- If $R(g) = g(X) \subseteq D(f)$, then $D(f \circ g) = X$.
- If $R(g) \not\subseteq D(f)$, then $D(f \circ g)$ is the largest set $A \subseteq X$ such that $g(A) \subseteq D(f)$.
- In general, $f \circ g \neq g \circ f$.

Example

Let $f(x) = 3x + 4$ and let $g(x) = \sqrt{x^2 - 1}$. Find

(a) $D(f \circ g)$ and $(f \circ g)(x)$.

(b) $D(g \circ f)$ and $(g \circ f)(x)$.

sol. $D(f) = \mathbb{R},\ R(f) = \mathbb{R}$ and $D(g) = (-\infty, -1] \cup [1, \infty),\ R(g) = [0, \infty)$.

(a) $\because g(X) = R(g) \subseteq D(f) \ \Rightarrow\ f \circ g$ is defined on X.

$\therefore D(f \circ g) = D(g) = (-\infty, -1] \cup [1, \infty)$ and

$$(f \circ g)(x) = f(g(x)) = 3g(x) + 4 = 3\sqrt{x^2 - 1} + 4.$$

(b) $\because R(f) \not\subseteq D(g)$, we need to find $A \subseteq X$ s.t. $f(A) \subseteq D(g)$

That is, find x s.t. $3x + 4 \geq 1$ or $3x + 4 \leq -1$.

$\therefore D(g \circ f) = A = (-\infty, -5/3] \cup [-1, \infty)$ and

$$(g \circ f)(x) = g(f(x)) = \sqrt{f(x)^2 - 1} = \sqrt{9x^2 + 24x + 15}.$$

Odd and Even Functions

Definition

- A function $y = f(x)$ is called even(偶函數) if $f(-x) = f(x)$.
- A function $y = f(x)$ is called odd(奇函數) if $f(-x) = -f(x)$.

Remark

- Let $y = f(x)$ be an even function and (x, y) is on the graph of f. Then $(-x, y)$ is also on the graph of f. That is, the graph of f is symmetric to y-axis.
- Let $y = f(x)$ be an odd function and (x, y) is on the graph of f. Then $(-x, -y)$ is also on the graph of f. That is, the graph of f is symmetric to the origin.

Example

Determine whether each function is even, odd, or neither.

1. $f(x) = x^2$.

   $\because f(-x) = (-x)^2 = x^2 = f(x)$. $\quad \therefore f(x)$ is an even function.

2. $f(x) = x^3$.

   $\because f(-x) = (-x)^3 = -x^3 = -f(x)$. $\quad \therefore f(x)$ is an odd function.

3. $f(x) = x^3 + x^2$.

   $\because f(-x) = (-x)^3 + (-x)^2 = -x^3 + x^2$

   $\Rightarrow f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. $\quad \therefore f(x)$ is neither even nor odd.

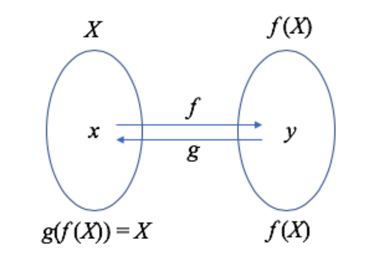
Inverse Functions

Definition

Let $f : X \mapsto Y$ be a function. Then f has an inverse if there exists a function $g : f(X) \mapsto X$ s.t.

$$f\,(g(y)) = y,\ \forall y \in f(X) \quad \& \quad g\,(f(x)) = x,\ \forall x \in X.$$

In this case, g is called the inverse function of f and is denoted by f^{-1}.

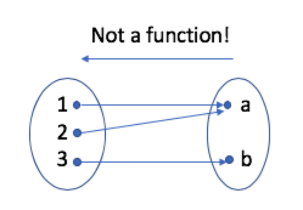


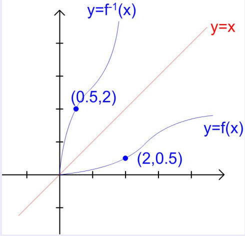
Remarks

- Let $f : X \mapsto Y$ have an inverse. Then $f^{-1} : f(X) \mapsto X$ and satisfies

$$f(f^{-1}(x)) = x, \ \forall x \in f(X) \quad \& \quad f^{-1}(f(x)) = x, \ \forall x \in X.$$

- $f^{-1}(x) \neq \dfrac{1}{f(x)}$. Here f^{-1} is just the label for the inverse function of f.
- f has an inverse if and only if f is $1-1$.

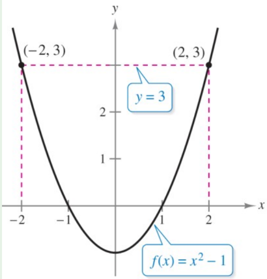




- The graph of f and f^{-1} are mirror images of each other w.r.t. (with respect to) the line $y = x$.

Example (A function without an inverse)

Show that the function $f(x) = x^2 - 1$ has no inverse function.



sol. $\because f(2) = 3 = f(-2) \Rightarrow f$ is not $1-1 \Rightarrow f$ has no inverse.

How to find f^{-1} algebraically?

To find the inverse of a function f algebraically:

1) Let $y = f^{-1}(x)$. ($\therefore f(y) = f(f^{-1}(x)) = x$.)
2) Solve $f(y) = x$ for y.
3) Check the domain of f^{-1}(= the range of f) and replace y by $f^{-1}(x)$.

Example

Find the inverse of $f(x) = 2x - 3$

sol. Let $y = f^{-1}(x)$ be the inverse of f.

$$\therefore f(y) = f(f^{-1}(x)) = x \;\Rightarrow\; 2y - 3 = x \;\Rightarrow\; y = \frac{x+3}{2}.$$

$$\because R(f) = \mathbb{R}. \quad \therefore f^{-1}(x) = y = \frac{x+3}{2},\ \forall x \in D(f^{-1}) = R(f) = \mathbb{R}.$$

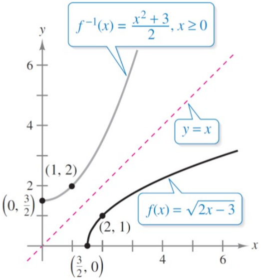
Example

Find the inverse function of $f(x) = \sqrt{2x-3}$

sol. Let $y = f^{-1}(x)$ be the inverse function of f.

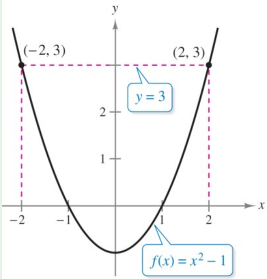
$$\therefore f(y) = f(f^{-1}(x)) = x \Rightarrow \sqrt{2y-3} = x \Rightarrow y = \frac{x^2+3}{2}.$$

$$\because R(f) = [0,\infty). \quad \therefore f^{-1}(x) = y = \frac{x^2+3}{2},\ \forall x \in D(f^{-1}) = R(f) = [0,\infty).$$



Example (A function without a inverse function)

Show that the function $f(x) = x^2 - 1$ has no inverse function.



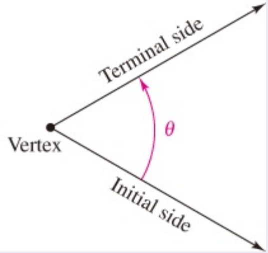
sol. $\because f(y) = x \;\Rightarrow\; y^2 - 1 = x \;\Rightarrow\; y = \pm\sqrt{x+1}$.

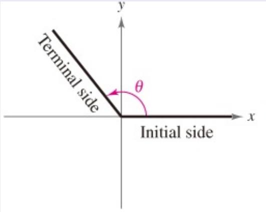
$\therefore$ y can not be defined as a function of $x \;\Rightarrow\; f$ has no inverse.

Angles and Degree Measure

Definition

1. An angle has three parts: an initial side, a terminal side, and a vertex.
2. An angle is in standard position when its initial side coincides with the positive x-axis and its vertex is at the origin.





3. Positive angles are measured counterclockwise beginning with the initial side. Negative angles are measured clockwise.
4. Angles having the same initial and terminal sides are called coterminal.

Remark

θ and ϕ are coterminal. Then there exists $k \in \mathbb{Z}$ such that $\theta = \phi + k \cdot 360^\circ$.

Example

1. 750° and 30° are coterminal. $(\because 750^\circ = 30^\circ + 2 \cdot 360^\circ)$.
2. -120° and 600° are coterminal. $(\because -120^\circ = 600^\circ + (-2) \cdot 360^\circ)$.

Radian measure

Definition

Let θ be the central angle of a circular sector of radius 1.

The radian measure of θ is defined to be the length of the arc of the sector(扇形) of radius 1.

Remark

$\because$ the circumference of the unit circle is $2\pi \Rightarrow 360^\circ = 2\pi$ radians.

$\therefore\ 1^\circ = \dfrac{\pi}{180}$ (radians) and 1 (radian) $= \dfrac{180^\circ}{\pi}$.

Example

1. $135^\circ = \dfrac{3\pi}{4}$ (radians); $\quad -90^\circ = -\dfrac{\pi}{2}$.
2. $\dfrac{11\pi}{6} = 330^\circ$; $\quad -\dfrac{5\pi}{4} = -225^\circ$.

The Trigonometric Functions

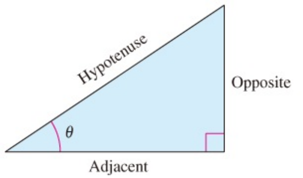
Definition

- Right triangle definition $\left(0 < \theta < \dfrac{\pi}{2}\right)$:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}; \quad \cos\theta = \frac{\text{adj}}{\text{hyp}}; \quad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\cot\theta = \frac{\text{adj}}{\text{opp}}; \quad \sec\theta = \frac{\text{hyp}}{\text{adj}}; \quad \sin\theta = \frac{\text{hyp}}{\text{opp}}$$

.

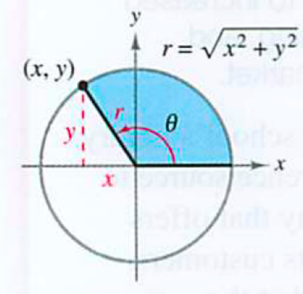


The Trigonometric Functions

Definition

- Circular function definition:

$$\sin\theta = \frac{y}{r};\quad \cos\theta = \frac{x}{r};\quad \tan\theta = \frac{y}{x};\quad \cot\theta = \frac{x}{y};\quad \sec\theta = \frac{r}{x};\quad \csc\theta = \frac{r}{y}.$$

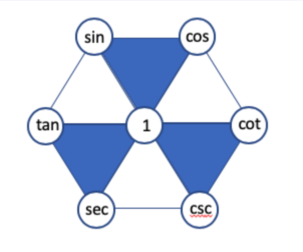


Formulas

Formulas I

1. $\sin\theta\csc\theta = 1; \quad \cos\theta\sec\theta = 1; \quad \tan\theta\cot\theta = 1.$
2. $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \sin\theta\sec\theta; \quad \cot\theta = \dfrac{\cos\theta}{\sin\theta} = \cos\theta\csc\theta.$
3. (Pythagorean Identities)

   $\sin^2\theta + \cos^2\theta = 1; \quad \tan^2\theta + 1 = \sec^2\theta; \quad \cot^2\theta + 1 = \csc^2\theta.$



Formulas II

(Reduction Formulas)

1. $\sin(-\theta) = \sin(2\pi - \theta) = -\sin\theta$; $\cos(-\theta) = \cos(2\pi - \theta) = \cos\theta$; $\tan(-\theta) = \tan(2\pi - \theta) = -\tan\theta$.
2. $\sin\left(\frac{\pi}{2} \pm \theta\right) = \cos\theta$; $\cos\left(\frac{\pi}{2} \pm \theta\right) = \mp\sin\theta$; $\tan\left(\frac{\pi}{2} \pm \theta\right) = \mp\cot\theta$.
3. $\sin(\pi \pm \theta) = \mp\sin\theta$; $\cos(\pi \pm \theta) = -\cos\theta$; $\tan(\pi \pm \theta) = \pm\tan\theta$.
4. $\sin\left(\frac{3\pi}{2} \pm \theta\right) = -\cos\theta$; $\cos\left(\frac{3\pi}{2} \pm \theta\right) = \pm\sin\theta$; $\tan\left(\frac{3\pi}{2} \pm \theta\right) = \mp\cot\theta$.

Formulas III

1. (Sum and difference formulas)

   (a) $\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$.

   (b) $\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$.

   (c) $\tan(\theta \pm \phi) = \dfrac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$.

2. (Multiple-angle formulas)

   (a) $\sin 2\theta = 2\sin\theta\cos\theta$;

   $\cos 2\theta = 2\cos^2\theta - 1 = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$.

   (b) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$; $\quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

3. (Half-angle formulas) $\sin^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{2}$; $\quad \cos^2\dfrac{\theta}{2} = \dfrac{1+\cos\theta}{2}$.

Example

1. Let $(1, -\sqrt{3})$ be a point on the terminal side of θ.

   $$\therefore r = \sqrt{1^2 + (-\sqrt{3})^2} = 2 \;\Rightarrow\; \sin\theta = -\frac{\sqrt{3}}{2},\ \cos\theta = \frac{1}{2},\ \tan\theta = -\sqrt{3}.$$

2.

θ	$0\ (0^\circ)$	$\frac{\pi}{6}\ (30^\circ)$	$\frac{\pi}{4}\ (45^\circ)$	$\frac{\pi}{3}\ (60^\circ)$	$\frac{\pi}{2}\ (90^\circ)$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
θ	$\pi\ (180^\circ)$	$\frac{3\pi}{2}\ (270^\circ)$			
$\sin\theta$	0	-1			
$\cos\theta$	-1	0			
$\tan\theta$	0	undefined			

Example

1. $$\sin\frac{4\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\sqrt{3}.$$

   $$\cos\frac{5\pi}{6} = \cos\left(\pi - \frac{\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}.$$

   $$\tan\frac{7\pi}{4} = \tan\left(2\pi - \frac{\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right) = -\tan\frac{\pi}{4} = -1.$$

2. $$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

   $$\tan\frac{5\pi}{12} = \tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = 2 + \sqrt{3}.$$

Example

1. Find θ such that $\sin\theta = \frac{1}{2}$.

   $\therefore\ \theta = \frac{\pi}{6} + 2k\pi$ or $\frac{5\pi}{6} + 2k\pi$, where $k \in \mathbb{Z}$.

2. Let $0 \leq \theta < 2\pi$. Solve $\cos 2\theta = 2 - 3\sin\theta$.

   $\because 1 - 2\sin^2\theta = 2 - 3\sin\theta \ \Rightarrow\ 2\sin^2\theta - 3\sin\theta + 1 = 0.$

   $\therefore \sin\theta = \frac{1}{2}$ or $1 \ \Rightarrow\ \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}.$

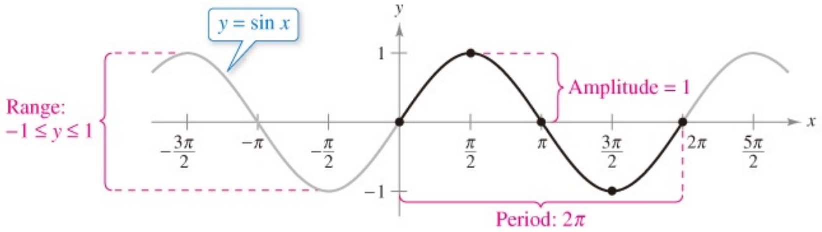
Graphs of Trigonometric Functions

Definition

A function $f(x)$ is called a periodic function if there exists $p > 0$ such that $f(x+p) = f(x)$, for all $x \in D(f)$.

In this case, the minimum of those positive p's is called the period of f.

The graph of $y = \sin x$:



Properties

1.

	$\sin x$	$\cos x$	$\tan x$
Domain	$\mathbb{R}$	$\mathbb{R}$	$x \neq \frac{\pi}{2} + n\pi$
Range	$[-1, 1]$	$[-1, 1]$	$\mathbb{R}$
Period	2π	2π	π
	$\cot x$	$\sec x$	$\csc x$
Domain	$x \neq n\pi$	$x \neq \frac{\pi}{2} + n\pi$	$x \neq n\pi$
Range	$\mathbb{R}$	$(-\infty, -1] \cup [1, \infty)$	$(-\infty, -1] \cup [1, \infty)$
Period	π	2π	2π

2.

Function	Period	Amplitude
$y = a\sin bx$ or $y = a\cos bx$	$\frac{2\pi}{\lvert b\rvert}$	$\lvert a\rvert$
$y = a\tan bx$ or $y = a\cot bx$	$\frac{\pi}{\lvert b\rvert}$	Not applicable
$y = a\sec bx$ or $y = a\csc bx$	$\frac{2\pi}{\lvert b\rvert}$	Not applicable