

1. (12%) Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out)

(a) $\lim_{x \rightarrow 3} \frac{x^2 + 14x - 51}{x^3 - 5x^2 + 4x + 6}$

(b) $\lim_{x \rightarrow 0} \frac{\sqrt{3+x} - \sqrt{3}}{x}$

(c) $\lim_{x \rightarrow 0} x \left(\cos 2x + \cos \frac{1}{x} \right)$

(d) $\lim_{x \rightarrow 2} \frac{2x+6}{x-2}$

Ans:

(a) $\lim_{x \rightarrow 3} \frac{x^2 + 14x - 51}{x^3 - 5x^2 + 4x + 6} = \lim_{x \rightarrow 3} \frac{(x-3)(x+17)}{(x-3)(x^2 - 2x - 2)} = \lim_{x \rightarrow 3} \frac{(x+17)}{(x^2 - 2x - 2)} = \lim_{x \rightarrow 3} \frac{(3+17)}{(3^2 - 2 \cdot 3 - 2)} = 20$

(Dividing out technique)

(b) $\lim_{x \rightarrow 0} \frac{\sqrt{3+x} - \sqrt{3}}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{3+x} - \sqrt{3}}{x} \cdot \frac{\sqrt{3+x} + \sqrt{3}}{\sqrt{3+x} + \sqrt{3}} = \lim_{x \rightarrow 0} \frac{3+x-3}{x(\sqrt{3+x} + \sqrt{3})}$
 $= \lim_{x \rightarrow 0} \frac{1}{\sqrt{3+x} + \sqrt{3}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$

(Rationalizing technique)

(c) For any x , $-2 \leq \cos 2x + \cos \frac{1}{x} \leq 2 \Rightarrow -2|x| \leq x \left(\cos 2x + \cos \frac{1}{x} \right) \leq 2|x|$,

According to Squeeze theorem, $\lim_{x \rightarrow 0} x \left(\cos 2x + \cos \frac{1}{x} \right) = 0$

Notice $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$ $\lim_{x \rightarrow 0} \frac{2x}{\sin(5x)} = \frac{2}{5}$

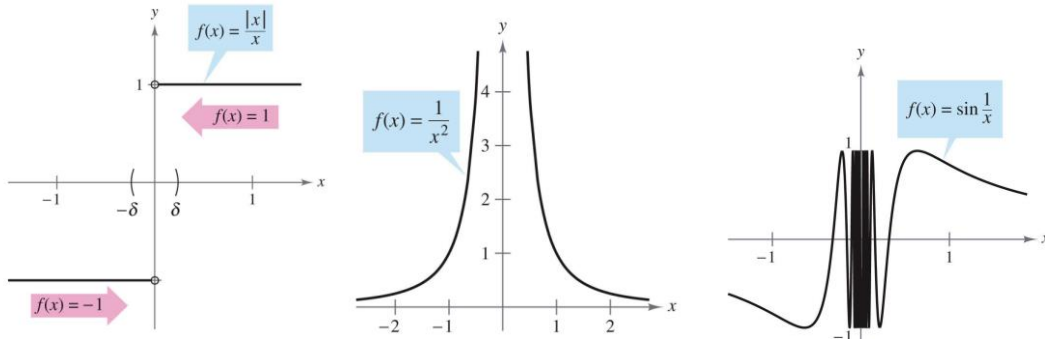
Theorem 1.8 (The Squeeze Theorem)

If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c , except possibly at c itself, and if

$$\lim_{x \rightarrow c} h(x) = L = \lim_{x \rightarrow c} g(x)$$

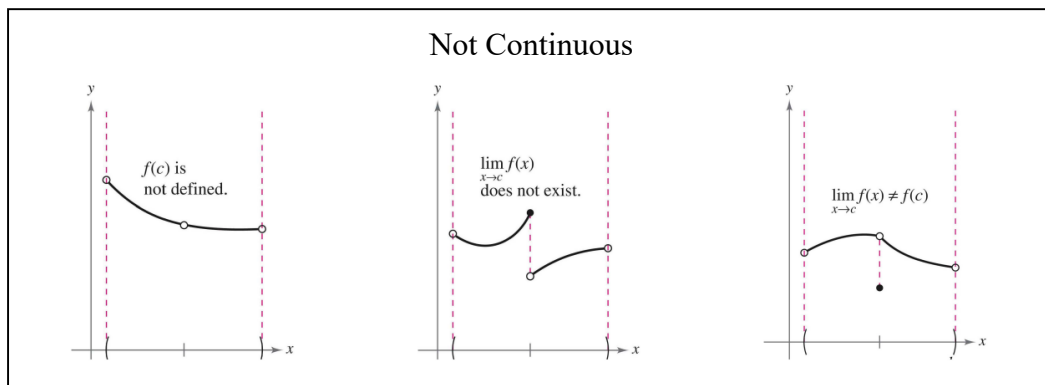
then $\lim_{x \rightarrow c} f(x)$ exists and is equal to L .

(d) $\lim_{x \rightarrow 2^+} \frac{2x+6}{x-2} = \infty$ and $\lim_{x \rightarrow 2^-} \frac{2x+6}{x-2} = -\infty$. Therefore, the limit does not exist



2. (8%) Determine all values of the constant a such that the following function is continuous at $x = 0$. (感謝同學提醒，原題因 $x = 0$ 未定義故送分，應改為以下型式)

$$f(x) = \begin{cases} a^2 - 2, & x \leq 0 \\ \frac{ax}{\tan x}, & x > 0 \end{cases}$$



Ans:

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (a^2 - 2) = a^2 - 2$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{ax}{\tan x} = \lim_{x \rightarrow 0^+} \frac{x}{\tan x} \lim_{x \rightarrow 0^+} a = a \quad (\text{Remember the proof at 69})$$

$$\text{Thus, } a^2 - 2 = a \rightarrow (a - 2)(a + 1) = 0 \rightarrow a = -1, 2$$

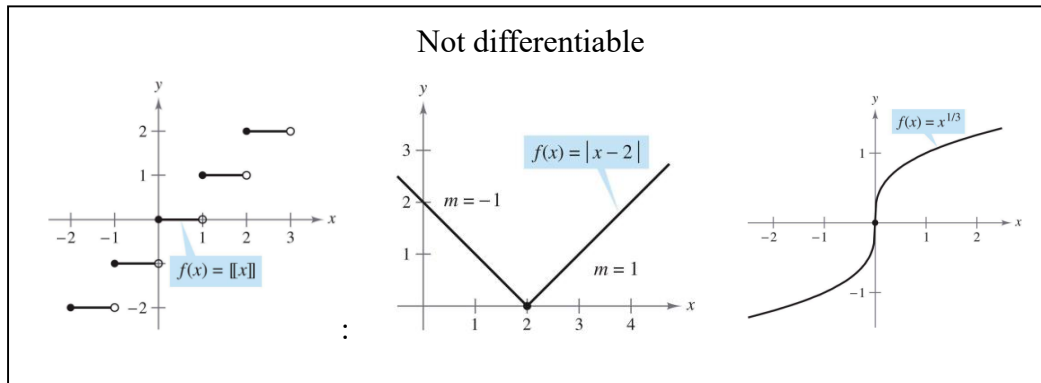
Definition 2.2 (The derivative of a function)

The derivative of f at x is given by

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x .

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$



3. (10%) Proof that there is only one intersect point between $f(x) = 5x^3 + 2x^2 + 4x + 1$ and $g(x) = 2x^2 + \cos x$. (Hint: use the mean value theorem)

Theorem 1.13 (The Intermediate Value Theorem)

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

Theorem 3.3 (Rolle's Theorem)

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) . If

$$f(a) = f(b)$$

then there is at least one number c in (a, b) such that $f'(c) = 0$.

Theorem 3.4 (The Mean Value Theorem)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Ans:

Let $F(x) = f(x) - g(x)$, clearly $F(0) = 0$. Using proof by contradiction, assume there exist a such that $F(a) = 0, a \neq 0$. According to Mean value theorem, $\exists c \in$

$(a, 0)$ such that $F'(c) = \frac{F(a) - F(0)}{a - 0} = 0$, contradict. ($F'(x) = 15x^2 + 4 + \sin x >$

0). Therefore, there is only one intersect point.

Related questions:

Exercise in 1.4

Exercise in 3.2

4. (12%) Remember that you can solve the derivative using the definition or the differentiation rule for the following question.

(a) Given $f(x) = \frac{x}{(x+1)(x+2)\dots(x+2021)}$, what is the value of $f'(0)$?

(b) Find the derivative of $f(x) = \frac{x^3+5x+3}{x^2-1}$

(c) Find the derivative of $f(x) = \sin(\sqrt{\cot(5\pi x)})$

(d) Find the following limit. $\lim_{x \rightarrow 0} \frac{\sin(\sqrt{3+x}) - \sin(\sqrt{3})}{x}$

Ans:

(a) $f'(0) = \lim_{\Delta x \rightarrow 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{\Delta x}{(\Delta x+1)(\Delta x+2)\dots(\Delta x+2021)} - \frac{0}{(0+1)(0+2)\dots(0+2021)}}{\Delta x} = \frac{1}{2021!}$

(b) $f'(x) = \frac{(x^2-1)(3x^2+5) - (x^3+5x+3)(2x)}{(x^2-1)^2} = \frac{x^4 - 8x^2 - 6x - 5}{(x^2-1)^2}$

(c) $y = \sin((\cot(5\pi x))^{\frac{1}{2}})$, $y' =$

$$\cos(\cot(5\pi x)^{\frac{1}{2}}) \left[\frac{1}{2} \cot(5\pi x)^{-\frac{1}{2}} (-\csc^2(5\pi x))(5\pi) \right] = -\frac{5\pi \cos(\sqrt{\cot(5\pi x)}) \csc^2(5\pi x)}{2\sqrt{\cot(5\pi x)}}$$

(d) Let $f(x) = \sin(\sqrt{x})$, then the limit is the derivative of $f(x)$ at $x = 3$. Which is

$$\frac{\cos(\sqrt{3})}{2\sqrt{3}}$$

Theorem 2.10 (The Chain Rule)

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x , then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx} [f(g(x))] = f'(g(x))g'(x).$$

5. (10%) Given $x^2y^3 - 5xy^2 - 4y = 4$, find the tangent line at (3,2)

Ans:

$$(x^2y^3 - 5xy^2 - 4y = 4)' = (4)'$$

$$y' = \frac{-7}{11}$$

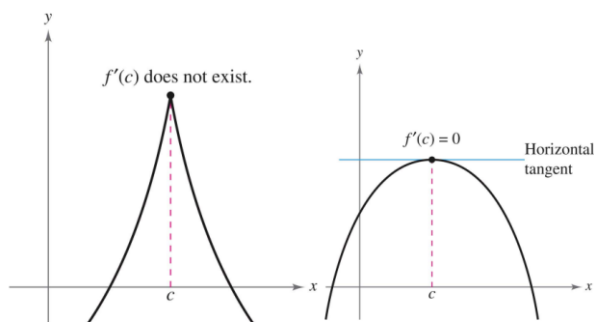
The tangent line is $(y - 2) = \frac{-7}{11}(x - 3)$

6. (10%) Let $f(x) = \frac{2x^2}{x^2-1}$

- Find the critical numbers and the possible points of inflection of $f(x)$
- Find the open intervals on which f is increasing or decreasing
- Find the open intervals of concavity
- Find all the asymptotes (Both vertical and horizontal)
- Sketch the graph of $f(x)$

Definition 3.3 (Critical number)

Let f be defined at c . If $f'(c) = 0$ or if f is not differentiable at c , then c is a critical number of f .

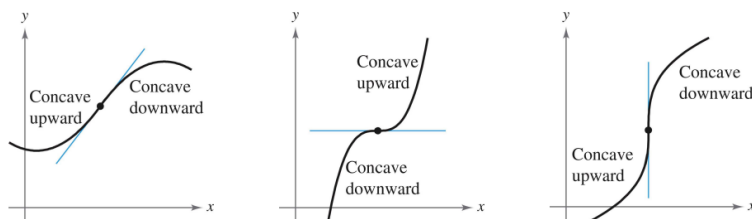


Definition 3.6 (Point of inflection)

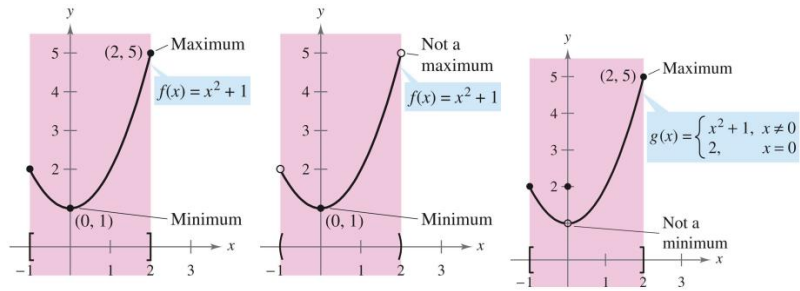
Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a point of inflection of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point.

Theorem 3.8 (Points of inflection)

If $(c, f(c))$ is a point of inflection of the graph of f , then either $f''(c) = 0$ or f'' does not exist at $x = c$.



Ans: $f(x) = \frac{2x^2}{x^2-1}, f'(x) = \frac{-4x}{(x^2-1)^2}, f''(x) = \frac{12x^2+4}{(x^2-1)^3}$



(a) The critical numbers are $x = 0$

x	$f(x)$	$f'(x)$	$f''(x)$	Graph
$-\infty < x < -1$		+	+	Increasing, upward
-1	Undef.	Undef.	Undef.	Vertical asymptote
$-1 < x < 0$		+	-	Increasing, downward
0	0	0	-4	Relative maximum
$0 < x < 1$		-	-	Decreasing, downward
1	Undef.	Undef.	Undef.	Vertical asymptote
$1 < x < \infty$		-	+	Decreasing, upward

Possible points of inflection: None

(b) Increasing $(-\infty, -1), (-1, 0)$. Decreasing $(0, 1), (1, \infty)$.

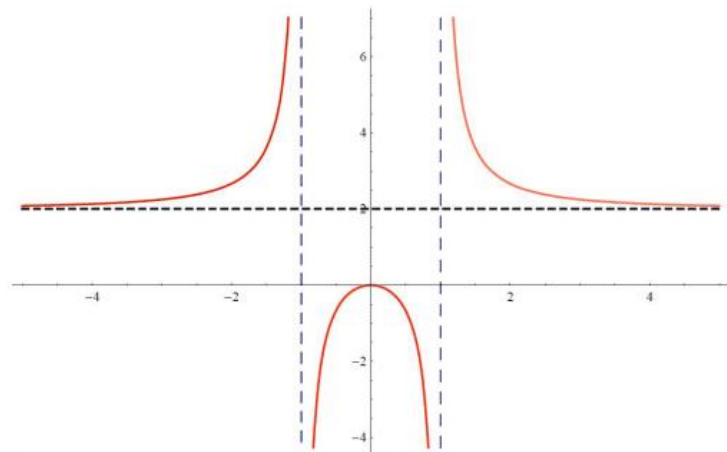
(c) Upward: $(-\infty, -1), (1, \infty)$. Downward $(-1, 1)$

(d) Since $\lim_{x \rightarrow \pm\infty} f(x) = 2 \rightarrow$ Horizontal asymptote $y = 2$

Since $\lim_{x \rightarrow 1^+} f(x) = \infty$ (or $\lim_{x \rightarrow 1^-} f(x) = -\infty$), vertical asymptote $x = 1$

Since $\lim_{x \rightarrow -1^+} f(x) = -\infty$ (or $\lim_{x \rightarrow -1^-} f(x) = \infty$), vertical asymptote $x = -1$

(e) Graph



7. (10%) Prove that $|\tan x - \tan y| \geq |x - y|$ for all $x, y \in (-\frac{\pi}{2}, \frac{\pi}{2})$ (Hint use mean value theorem)

Theorem 3.4 (The Mean Value Theorem)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Ans: Note the inequality holds for $x = y$

WLOG, assume $x < y$. (absolute value)

Let $f(t) = \tan(t)$. By the mean value theorem, $\exists c \in (x, y)$ such that $f'(c) = \frac{f(x) - f(y)}{x - y}$.

Since $f'(t) = \sec^2(t) \rightarrow f'(c) = \sec^2(c) \rightarrow \frac{|\tan x - \tan y|}{|x - y|} = |\sec^2(c)|$

$|\tan x - \tan y| = |\sec^2(c)| |x - y|$, since $|\sec^2(c)| \geq 1$, we have $|\tan x - \tan y| \geq |x - y|$

8. (12%)

(a) $\int \frac{1+x+x^2}{\sqrt{x}} dx$

(b) $\int 6t - \csc^2 t dt$

(c) Evaluate $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} (1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}})$ (Hint: use Riemann sum and the definition of the definite integral)

(d) $\int_{-\pi}^{\pi} \frac{x^3 \cos x}{1+x^4} dx$

Ans:

(a) $\int \frac{1+x+x^2}{\sqrt{x}} dx = 2\sqrt{x} + \frac{2x^{3/2}}{3} + \frac{2x^{5/2}}{5} + C$

(b) $\int 6t - \csc^2 t dt = 3t^2 + \cot(t) + C$

Definition 4.4 (Riemann sum)

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

where Δx_i is the width of the i th subinterval. If c_i is any point in the i th subinterval $[x_{i-1}, x_i]$, then the sum

$$\sum_{i=1}^n f(c_i) \Delta x_i, \quad x_{i-1} \leq c_i \leq x_i$$

is called a Riemann sum of f for the partition Δ .

Definition 4.5 (Definite integral)

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$$

exists (as described above), then f is said to be integrable on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i = \int_a^b f(x) dx.$$

The limit is called the definite integral of f from a to b . The number a is the lower limit of integration, and the number b is the upper limit of integration.

$$(c) \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sqrt{n} + \frac{\sqrt{n}}{\sqrt{2}} + \frac{\sqrt{n}}{\sqrt{3}} + \dots + \frac{\sqrt{n}}{\sqrt{n}} \right) =$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{1}{\sqrt{\frac{1}{n}}} + \frac{1}{\sqrt{\frac{2}{n}}} + \frac{1}{\sqrt{\frac{3}{n}}} + \dots + \frac{1}{\sqrt{\frac{n}{n}}} \right) = \int_0^1 \frac{1}{\sqrt{x}} dx = 2x^{\frac{1}{2}} \Big|_0^1 = 2$$

Theorem 4.16 (Integration of even and odd functions)

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an even function, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.
2. If f is an odd function, then $\int_{-a}^a f(x) dx = 0$.

(d) Since it is an odd function, $\int_{-\pi}^{\pi} \frac{x^3 \cos x}{1+x^6} dx = 0$

9. (8%) Find $\frac{d}{dx} \int_{2x}^{x^2} \cos\sqrt{t} dt$ when $x > 0$. (Hint: Let $F(x) = \int_1^x \cos\sqrt{t} dt$ and use the fundamental theorem of calculus)

Theorem 4.9 (The Fundamental Theorem of Calculus)

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Theorem 4.11 (The Second Fundamental Theorem of Calculus)

If f is continuous on an open interval I containing a , then, for every x in the interval,

$$\frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x).$$

Ans: Let $F(x) = \int_1^x \cos\sqrt{t} dt$, since $\cos\sqrt{t}$ is continuous, by the fundamental theorem of calculus, $F'(x) = \cos\sqrt{x}$. Also $F(b) - F(a) = \int_a^b \cos\sqrt{t} dt$, $a, b \in \mathbb{R}$, therefore, when $x > 0$,

$$\begin{aligned} \frac{d}{dx} \int_{2x}^{x^2} \cos\sqrt{t} dt &= \frac{d}{dx} [F(x^2) - F(2x)] = F'(x^2)2x - F'(2x)2 \\ &= 2x\cos x - 2\cos\sqrt{2x} \end{aligned}$$

Related questions:

Exercise in chapter 4.4

10. (8%) Evaluate $\int_1^2 2x^2\sqrt{x^3+1}dx$

Theorem 4.13 (Antidifferentiation of a composite function)

Let g be a function whose range is an interval I , and let f be a function that is continuous on I . If g is differentiable on its domain and F is an antiderivative of f on I , then

$$\int f(g(x))g'(x) dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x) dx$ and

$$\int f(u) du = F(u) + C.$$

Guidelines for making a change of variables

- 1 Choose a substitution $u = g(x)$. Usually, it is best to choose the inner part of a composite function, such as a quantity raised to a power.
- 2 Compute $du = g'(x) dx$.
- 3 Rewrite the integral in terms of the variable u .
- 4 Find the resulting integral in terms of u .
- 5 Replace u by $g(x)$ to obtain an antiderivative in terms of x .
- 6 Check your answer by differentiating.

Theorem 4.15 (Change of variables for definite integrals)

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g , then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

Ans:

Let $u = x^3 + 1, du = 3x^2 dx$

$$\int_1^2 2x^2\sqrt{x^3+1}dx = 2 \frac{1}{3} \int_1^2 (x^3+1)^{1/2} (3x^2) dx = \frac{4}{9} [(x^3+1)^{3/2}]_1^2 = 12 - \frac{8}{9}\sqrt{2}$$