# Chapter 8 Integration Techniques and Improper Integrals

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**Example 1 (A comparison of three similar integrals)**

Find each integral.

**a.** $\int \frac{4}{x^2+9}\,\mathrm{d}x$ **b.** $\int \frac{4x}{x^2+9}\,\mathrm{d}x$ **c.** $\int \frac{4x^2}{x^2+9}\,\mathrm{d}x$

**a.** Use the Arctangent Rule and let $u = x$ and $a = 3$.

$$\int \frac{4}{x^2+9}\,\mathrm{d}x = 4\int \frac{1}{x^2+3^2}\,\mathrm{d}x = 4\left(\frac{1}{3}\arctan\frac{x}{3}\right) + C$$
$$= \frac{4}{3}\arctan\frac{x}{3} + C$$

**b.** Here the Arctangent Rule does not apply because the numerator contains a factor of $x$. Consider the Log Rule and let $u = x^2 + 9$.

- Then $\mathrm{d}u = 2x\,\mathrm{d}x$, and you have

$$\int \frac{4x}{x^2+9}\,\mathrm{d}x = 2\int \frac{2x}{x^2+9}\,\mathrm{d}x = 2\int \frac{\mathrm{d}u}{u}$$
$$= 2\ln|u| + C = 2\ln(x^2+9) + C$$

**c.** Because the degree of the numerator is equal to the degree of the denominator, you should first use division to rewrite the improper rational function as the sum of a polynomial and a proper rational function.

- $$\begin{aligned}\int \frac{4x^2}{x^2+9}\,\mathrm{d}x &= \int \left(4 - \frac{36}{x^2+9}\right)\mathrm{d}x = \int 4\,\mathrm{d}x - 36\int \frac{1}{x^2+9}\,\mathrm{d}x \\ &= 4x - 36\left(\frac{1}{3}\arctan\frac{x}{3}\right) + C = 4x - 12\arctan\frac{x}{3} + C.\end{aligned}$$

■

## Example 2 (Using two basic rules to solve a single integral)

Evaluate $\int_0^1 \frac{x+3}{\sqrt{4-x^2}}\,\mathrm{d}x$.

- Begin by writing the integral as the sum of two integrals.

- Then apply the Power Rule and the Arcsine Rule, as follows.

$$\begin{aligned}\int_0^1 \frac{x+3}{\sqrt{4-x^2}}\,dx &= \int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx + \int_0^1 \frac{3}{\sqrt{4-x^2}}\,dx \\ &= \frac{-1}{2}\int_0^1 (4-x^2)^{-1/2}(-2x)\,dx + 3\int_0^1 \frac{1}{\sqrt{2^2-x^2}}\,dx \\ &= \left[-(4-x^2)^{1/2} + 3\sin^{-1}\frac{x}{2}\right]_0^1 \\ &= \left(-\sqrt{3}+\frac{\pi}{2}\right) - (-2+0) \approx 1.839\end{aligned}$$

■

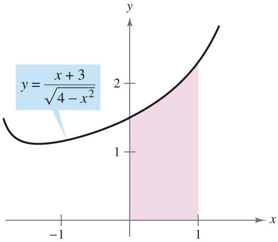


Figure 1: The area of the region is approximately 1.839.

## Example 3 (A substitution involving $a^2 - u^2$)

Find $\int \frac{x^2}{\sqrt{16-x^6}}\,\mathrm{d}x$.

- Because the radical in the denominator can be written in the form

$$\sqrt{a^2 - u^2} = \sqrt{4^2 - (x^3)^2}$$

  you can try the substitution $u = x^3$.

- Then $\mathrm{d}u = 3x^2\,\mathrm{d}x$, and you have

$$\int \frac{x^2}{\sqrt{16 - x^6}}\,\mathrm{d}x = \frac{1}{3}\int \frac{3x^2\,\mathrm{d}x}{\sqrt{4^2 - (x^3)^2}} = \frac{1}{3}\int \frac{\mathrm{d}u}{\sqrt{4^2 - u^2}}$$
$$= \frac{1}{3}\arcsin\frac{u}{4} + C = \frac{1}{3}\arcsin\frac{x^3}{4} + C. \qquad \blacksquare$$

## Example 4 (A disguised form of the Log Rule)

Find $\int \frac{1}{1+e^x}\,\mathrm{d}x$.

- The integral does not appear to fit any of the basic rules. However, the quotient form suggests the Log Rule.
- If you let $u = 1 + e^x$, then $\mathrm{d}u = e^x\,\mathrm{d}x$. You can obtain the required $\mathrm{d}u$ by adding and subtracting $e^x$ in the numerator, as follows.

$$\int \frac{1}{1+e^x}\,\mathrm{d}x = \int \frac{1+e^x-e^x}{1+e^x}\,\mathrm{d}x = \int \left(\frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x}\right)\mathrm{d}x$$

$$= \int 1\,\mathrm{d}x - \int \frac{e^x}{1+e^x}\,\mathrm{d}x = x - \ln(1+e^x) + C \qquad \blacksquare$$

## Example 5 (A disguised form of the Power Rule)

Find $\int (\cot x) \ln(\sin x)\, \mathrm{d}x$.

Let $u = \ln(\sin x)$. Then $\mathrm{d}u = \frac{\cos x}{\sin x}\, \mathrm{d}x = \cot x\, \mathrm{d}x$.

$$\begin{aligned}\int (\cot x) \ln(\sin x)\, \mathrm{d}x &= \int u\, \mathrm{d}u \\ &= \frac{u^2}{2} + C \\ &= \frac{1}{2}[\ln(\sin x)]^2 + C.\end{aligned}$$

■

## Example 6 (Using trigonometric identities)

Find $\int \tan^2 2x\, \mathrm{d}x$.

- Note that $\tan^2 u$ is not in the list of basic integration rules. However, $\sec^2 u$ is in the list.

- This suggests the trigonometric identity $\tan^2 u = \sec^2 u - 1$.
- If you let $u = 2x$, then $\mathrm{d}u = 2\,\mathrm{d}x$ and

$$\begin{aligned}
\int \tan^2 2x\,\mathrm{d}x &= \frac{1}{2}\int \tan^2 u\,\mathrm{d}u = \frac{1}{2}\int (\sec^2 u - 1)\,\mathrm{d}u \\
&= \frac{1}{2}\int \sec^2 u\,\mathrm{d}u - \frac{1}{2}\int \mathrm{d}u \\
&= \frac{1}{2}\tan u - \frac{u}{2} + C = \frac{1}{2}\tan 2x - x + C.
\end{aligned}$$

This section concludes with a summary of the common procedures for fitting integrands to the basic integration rules. ■

Table 1: Review of basic integration rules ($a > 0$)

| | |
|---|---|
| 1. $\int kf(u)\,\mathrm{d}u = k\int f(u)\,\mathrm{d}u$ | 2. $\int [f(u) \pm g(u)]\,\mathrm{d}u$ $= \int f(u)\,\mathrm{d}u \pm \int g(u)\,\mathrm{d}u$ |
| 3. $\int \mathrm{d}u = u + C$ | 4. $\int u^n\,\mathrm{d}u = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$ |
| 5. $\int \frac{\mathrm{d}u}{u} = \ln\lvert u\rvert + C$ | 6. $\int e^u\,\mathrm{d}u = e^u + C$ |
| 7. $\int a^u\,\mathrm{d}u = \left(\frac{1}{\ln a}\right) a^u + C$ | 8. $\int \sin u\,\mathrm{d}u = -\cos u + C$ |
| 9. $\int \cos u\,\mathrm{d}u = \sin u + C$ | 10. $\int \tan u\,\mathrm{d}u = -\ln\lvert\cos u\rvert + C$ |
| 11. $\int \cot u\,\mathrm{d}u = \ln\lvert\sin u\rvert + C$ | 12. $\int \sec u\,\mathrm{d}u = \ln\lvert\sec u + \tan u\rvert + C$ |
| 13. $\int \csc u\,\mathrm{d}u = -\ln\lvert\csc u + \cot u\rvert + C$ | 14. $\int \sec^2 u\,\mathrm{d}u = \tan u + C$ |
| 15. $\int \csc^2 u\,\mathrm{d}u = -\cot u + C$ | 16. $\int \sec u \tan u\,\mathrm{d}u = \sec u + C$ |
| 17. $\int \csc u \cot u\,\mathrm{d}u = -\csc u + C$ | 18. $\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$ |
| 19. $\int \frac{\mathrm{d}u}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$ | 20. $\int \frac{\mathrm{d}u}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{\lvert u\rvert}{a} + C$ |

Procedures for fitting integrands to basic integration

| Technique | Example |
|---|---|
| Expand (numerator). | $(1+e^x)^2 = 1 + 2e^x + e^{2x}$ |
| Separate numerator. | $\frac{1+x}{x^2+1} = \frac{1}{x^2+1} + \frac{x}{x^2+1}$ |
| Complete the square. | $\frac{1}{\sqrt{2x-x^2}} = \frac{1}{\sqrt{1-(x-1)^2}}$ |
| Divide improper rational function. | $\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$ |
| Add and subtract terms in numerator. | $\frac{2x}{x^2+2x+1} = \frac{2x+2-2}{x^2+2x+1} = \frac{2x+2}{x^2+2x+1} - \frac{2}{(x+1)^2}$ |
| Use trigonometric identities. | $\cot^2 x = \csc^2 x - 1$ |
| Multiply and divide by Pythagorean conjugate | $\frac{1}{1+\sin x} = \left(\frac{1}{1+\sin x}\right)\left(\frac{1-\sin x}{1-\sin x}\right) = \frac{1-\sin x}{1-\sin^2 x}$ $= \frac{1-\sin x}{\cos^2 x} = \sec^2 x - \frac{\sin x}{\cos^2 x}$ |

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# Integration by parts

- In this section you will study an important integration technique called integration by parts. This technique can be applied to a wide variety of functions and is particularly useful for integrands involving products of algebraic and transcendental functions.
- For instance, integration by parts works well with integrals such as

$$\int x \ln x \,\mathrm{d}x, \quad \int x^2 e^x \,\mathrm{d}x, \quad \text{and} \quad \int e^x \sin x \,\mathrm{d}x.$$

- Integration by parts is based on the formula for the derivative of a product

$$\frac{\mathrm{d}}{\mathrm{d}x}[uv] = uv' + vu'.$$

- If $u'$ and $v'$ are continuous, you can integrate both sides of this equation to obtain

$$uv = \int uv' \,\mathrm{d}x + \int vu' \,\mathrm{d}x = \int u \,\mathrm{d}v + \int v \,\mathrm{d}u.$$

## Theorem 8.1 (Integration by Parts)

*If $u$ and $v$ are functions of $x$ and have continuous derivatives, then*

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u = uv - \int vu'\,\mathrm{d}x.$$

Guidelines for integration by parts

1. Try letting $\mathrm{d}v$ be the most complicated portion of the integration rule. Then $u$ will be remaining factor(s) of the integrand.
2. Trying letting $u$ be the portion of the integrated whose derivative is a function simpler than $u$. Then $\mathrm{d}v$ will be the remaining factor(s) of the integrand.

Note that $\mathrm{d}v$ always includes the $\mathrm{d}x$ of the original integrand.

## Example 1 (Integration by parts)

Find $\int xe^x \, \mathrm{d}x$.

- To apply integration by parts, you need to write the integral in the form $\int u \, \mathrm{d}v$.
- There are several ways to do this.

$$\int \underbrace{(x)}_{u} \underbrace{(e^x \, \mathrm{d}x)}_{\mathrm{d}v}, \quad \int \underbrace{(e^x)}_{u} \underbrace{(x \, \mathrm{d}x)}_{\mathrm{d}v}, \quad \int \underbrace{(1)}_{u} \underbrace{(xe^x \, \mathrm{d}x)}_{\mathrm{d}v}, \quad \int \underbrace{(xe^x)}_{u} \underbrace{(\,\mathrm{d}x)}_{\mathrm{d}v}$$

- The guidelines suggest the first option because the derivative of $u = x$ is simpler than $x$, and $\mathrm{d}v = e^x \, \mathrm{d}x$ is the most complicated portion of the integrand that fits a basic integration formula.

$$\begin{aligned} \mathrm{d}v = e^x \, \mathrm{d}x \quad &\Longrightarrow \quad v = \int \mathrm{d}v = \int e^x \, \mathrm{d}x = e^x \\ u = x \quad &\Longrightarrow \quad \mathrm{d}u = \mathrm{d}x \end{aligned}$$

- Now, integration by parts produces

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u \quad \int xe^x\,\mathrm{d}x = xe^x - \int e^x\,\mathrm{d}x = xe^x - e^x + C.$$

- To check this, differentiate $xe^x - e^x + C$ to see that you obtain the original integrand. ■

**Example 2 (Integration by parts)**

Find $\int x^2 \ln x\,\mathrm{d}x$.

- Let $\mathrm{d}v = x^2\,\mathrm{d}x$.

$$\mathrm{d}v = x^2\,\mathrm{d}x \quad \Longrightarrow \quad v = \int \mathrm{d}v = \int x^2\,\mathrm{d}x = \frac{x^3}{3}$$

$$u = \ln x \quad \Longrightarrow \quad \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$$

- Integration by parts now produces

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

$$\begin{aligned}\int x^2 \ln x\,\mathrm{d}x &= \frac{x^3}{3}\ln x - \int \left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right)\mathrm{d}x \\ &= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,\mathrm{d}x = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C. \qquad \blacksquare\end{aligned}$$

**Example 3 (An integrand with a single term)**

Evaluate $\int_0^1 \sin^{-1} x\,\mathrm{d}x$.

- Let $\mathrm{d}v = \mathrm{d}x$.

$$\begin{aligned}\mathrm{d}v = \mathrm{d}x \quad &\Longrightarrow \quad v = \int \mathrm{d}x = x \\ u = \sin^{-1} x \quad &\Longrightarrow \quad \mathrm{d}u = \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x\end{aligned}$$

- Integration by parts now produces

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u$$

$$\begin{aligned}\int \sin^{-1} x \, \mathrm{d}x &= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} \, \mathrm{d}x \\ &= x \sin^{-1} x + \frac{1}{2} \int \left(1-x^2\right)^{-1/2} (-2x) \, \mathrm{d}x \\ &= x \sin^{-1} x + \sqrt{1-x^2} + C.\end{aligned}$$

- Using this antiderivative, you can evaluate the definite integral as follows.

$$\int_0^1 \sin^{-1} x \, \mathrm{d}x = \left[x \sin^{-1} x + \sqrt{1-x^2}\right]_0^1 = \frac{\pi}{2} - 1 \approx 0.571.$$

  The area represented by this definite integral is shown in Figure 2.
- Alternative: If $y = \sin^{-1} x$, then $x = \sin y$. As shown in Figure 2,

$$\int_0^1 \sin^{-1} x \, \mathrm{d}x = (1)\left(\frac{\pi}{2}\right) - \int_0^{\pi/2} \sin y \, \mathrm{d}y = \frac{\pi}{2} - 1. \qquad \blacksquare$$

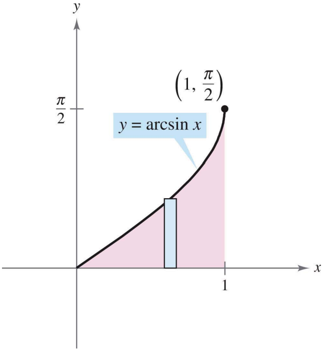


Figure 2: The area of the region is approximately 0.571.

A perspective: On integrating an inverse function

Let the function be strictly increasing and differentiable; the case of $f$ strictly decreasing is similar. Look at Figure 3. The area of the region marked $P$ is the area under the curve $x = f^{-1}(y)$ from $y = a$ to $y = b$. That is, we compute the area by interchanging the roles of $x$ and $y$ in the usual computation of area under a curve. Thus

$$\text{area of } P = \int_a^b f^{-1}(y)\,\mathrm{d}y.$$

The area of $Q$ is computed in the usual way:

$$\text{area of } Q = \int_{f^{-1}(a)}^{f^{-1}(b)} f(x)\,\mathrm{d}x.$$

Finally, the region marked $R$ is a rectangle, so

$$\text{area of } R = \text{base} \times \text{height } = f^{-1}(a) \times a = af^{-1}(a).$$

Now, the region $P + Q + R$ is a larger rectangle with base $f^{-1}(b)$ and height $b$. Thus,

$$\text{area of } P = \int_a^b f^{-1}(y)\,\mathrm{d}y = bf^{-1}(b) - af^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} f(x)\,\mathrm{d}x.$$

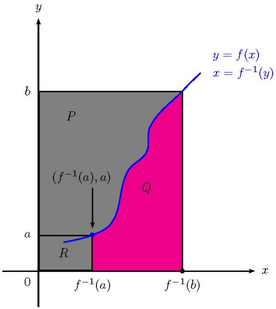


Figure 3: The area of $P = \int_a^b f^{-1}(y)\,\mathrm{d}y$. The area of $Q = \int_{f^{-1}(a)}^{f^{-1}(b)} f(x)\,\mathrm{d}x$. The area of $R = af^{-1}(a)$. The sum of the three areas is $bf^{-1}(b)$.

## Example 4 (Repeated use of integration by parts)

Find $\int x^2 \sin x \, \mathrm{d}x$.

- The factors $x^2$ and $\sin x$ are equally easy to integrate. However, the derivative of $x^2$ becomes simpler, whereas the derivative of $\sin x$ does not. So, you should let $u = x^2$.

$$\begin{aligned} \mathrm{d}v &= \sin x \, \mathrm{d}x \quad &\Longrightarrow \quad & v = \int \sin x \, \mathrm{d}x = -\cos x \\ u &= x^2 \quad &\Longrightarrow \quad & \mathrm{d}u = 2x \, \mathrm{d}x \end{aligned}$$

- Integration by parts produces

$$\int x^2 \sin x \, \mathrm{d}x = -x^2 \cos x + \int 2x \cos x \, \mathrm{d}x.$$

- To evaluate that integral, you can apply integration by parts again. This time, let $u = 2x$.

$$\begin{aligned} \mathrm{d}v = \cos x \, \mathrm{d}x \quad &\Longrightarrow \quad v = \int \cos x \, \mathrm{d}x = \sin x \\ u = 2x \quad &\Longrightarrow \quad \mathrm{d}u = 2 \, \mathrm{d}x \end{aligned}$$

- Now, integration by parts produce

$$\int 2x \cos x \, \mathrm{d}x = 2x \sin x - \int 2 \sin x \, \mathrm{d}x = 2x \sin x + 2 \cos x + C.$$

- Combining these two results, you can write

$$\int x^2 \sin x \, \mathrm{d}x = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

■

## Example 5 (Integration by parts)

Find $\int \sec^3 x \, \mathrm{d}x$.

- The most complicated portion of the integrand that can be easily integrated is $\sec^2 x$, so you should let $\mathrm{d}v = \sec^2 x \, \mathrm{d}x$ and $u = \sec x$.

$$\mathrm{d}v = \sec^2 x \, \mathrm{d}x \quad \Longrightarrow \quad v = \int \sec^2 x \, \mathrm{d}x = \tan x$$

$$u = \sec x \quad \Longrightarrow \quad \mathrm{d}u = \sec x \tan x \, \mathrm{d}x$$

- Integration by parts produces

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u$$

$$\begin{aligned}\int \sec^3 x \, \mathrm{d}x &= \sec x \tan x - \int \sec x \tan^2 x \, \mathrm{d}x \\ &= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, \mathrm{d}x \\ &= \sec x \tan x - \int \sec^3 x \, \mathrm{d}x + \int \sec x \, \mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
2\int \sec^3 x\,\mathrm{d}x &= \sec x \tan x + \int \sec x\,\mathrm{d}x \\
&= \sec x \tan x + \ln|\sec x + \tan x| + C \\
\int \sec^3 x\,\mathrm{d}x &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.
\end{aligned}
$$

■

Summary of common integrals using integration by parts

1. For integrals of the form

$$\int x^n e^{ax}\,\mathrm{d}x, \quad \int x^n \sin ax\,\mathrm{d}x, \quad \text{or} \quad \int x^n \cos ax\,\mathrm{d}x$$

   let $u = x^n$ and let $\mathrm{d}v = e^{ax}\,\mathrm{d}x$, $\sin ax\,\mathrm{d}x$, $\cos ax\,\mathrm{d}x$.

2. For integrals of the form

$$\int x^n \ln x\,\mathrm{d}x, \quad \int x^n \arcsin ax\,\mathrm{d}x, \quad \text{or} \quad \int x^n \arctan ax\,\mathrm{d}x$$

   let $u = \ln x$, $\arcsin ax$, or $\arctan x$ and let $\mathrm{d}v = x^n\,\mathrm{d}x$.

3. For integrals of the form

$$\int e^{ax} \sin bx\,\mathrm{d}x, \quad \text{or} \quad \int e^{ax} \cos bx\,\mathrm{d}x$$

   let $u = \sin bx$ or $\cos bx$ and let $\mathrm{d}v = e^{ax}\,\mathrm{d}x$.

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# Integrals involving powers of sine and cosine

- In this section you will study techniques for evaluating integrals of the form
$$\int \sin^m x \cos^n x \,\mathrm{d}x \quad \text{and} \quad \int \sec^m x \tan^n x \,\mathrm{d}x$$
where either $m$ or $n$ is a positive integer.
- To find antiderivatives for these forms, try to break them into combinations of trigonometric integrals to which you can apply the Power Rule.
- For instance, you can evaluate $\int \sin^5 x \cos x \,\mathrm{d}x$ with the Power Rule by letting $u = \sin x$. Then, $\mathrm{d}u = \cos x \,\mathrm{d}x$ and you have
$$\int \sin^5 x \cos x \,\mathrm{d}x = \int u^5 \,\mathrm{d}u = \frac{u^6}{6} + C = \frac{\sin^6 x}{6} + C.$$
- To break up $\int \sin^m x \cos^n x \,\mathrm{d}x$ into forms to which you can apply the Power Rule, use the following identities.
$$\sin^2 x + \cos^2 x = 1 \quad \sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Guidelines for evaluating integrals involving powers of sine and cosine

1. If the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines.

$$\int \sin^{\overbrace{2k+1}^{\text{Odd}}} x \cos^n x \, \mathrm{d}x = \int \overbrace{(\sin^2 x)^k}^{\text{Convert to cosines}} \cos^n x \overbrace{\sin x \, \mathrm{d}x}^{\text{Save for } \mathrm{d}u}$$

$$= \int (1-\cos^2 x)^k \cos^n x \sin x \, \mathrm{d}x$$

2. If the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines.

$$\int \sin^m x \cos^{\overbrace{2k+1}^{\text{Odd}}} x \, \mathrm{d}x = \int \sin^m x \overbrace{(\cos^2 x)^k}^{\text{Convert to cosines}} \overbrace{\cos x \, \mathrm{d}x}^{\text{Save for } \mathrm{d}u}$$

$$= \int \sin^m x (1-\sin^2 x)^k \cos x \, \mathrm{d}x$$

3. If the power of both the sine and cosine are even and nonnegative, make repeated use of the identities

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

to convert the integrand to odd powers of the cosine. Then proceed as in guideline 2.

**Example 1 (Power of sine is odd and positive)**

Find $\int \sin^3 x \cos^4 x \, \mathrm{d}x$.

- Because you expect to use the Power Rule with $u = \cos x$, save one sine factor to form $\mathrm{d}u$ and convert the remaining sine factors to cosines.

$$
\begin{aligned}
\int \sin^3 x \cos^4 x \,\mathrm{d}x &= \int \sin^2 x \cos^4 x (\sin x)\,\mathrm{d}x \\
&= \int (1-\cos^2 x)\cos^4 x \sin x \,\mathrm{d}x \\
&= \int (\cos^4 x - \cos^6 x)\sin x \,\mathrm{d}x \\
&= \int \cos^4 x \sin x \,\mathrm{d}x - \int \cos^6 x \sin x \,\mathrm{d}x \\
&= -\int \cos^4 x (-\sin x)\,\mathrm{d}x + \int \cos^6 x(-\sin x)\,\mathrm{d}x \\
&= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C \qquad \blacksquare
\end{aligned}
$$

## Example 2 (Power of cosine is odd and positive)

Find $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\,\mathrm{d}x$, as shown in Figure 4.

Because you expect to use the Power Rule with $u = \sin x$, save one cosine factor to form $\mathrm{d}u$ and convert the remaining cosine factors to sines.

$$\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\,\mathrm{d}x = \int_{\pi/6}^{\pi/3} \frac{\cos^2 x \cos x}{\sqrt{\sin x}}\,\mathrm{d}x = \int_{\pi/6}^{\pi/3} \frac{(1-\sin^2 x)(\cos x)}{\sqrt{\sin x}}\,\mathrm{d}x$$

$$= \int_{\pi/6}^{\pi/3} [(\sin x)^{-1/2}\cos x - (\sin x)^{3/2}\cos x\,\mathrm{d}x$$

$$= \left[\frac{(\sin x)^{1/2}}{1/2} - \frac{(\sin x)^{5/2}}{5/2}\right]_{\pi/6}^{\pi/3}$$

$$= 2\left(\frac{\sqrt{3}}{2}\right)^{1/2} - \frac{2}{5}\left(\frac{\sqrt{3}}{2}\right)^{5/2} - \sqrt{2} + \frac{\sqrt{32}}{80} \approx 0.239 \quad \blacksquare$$

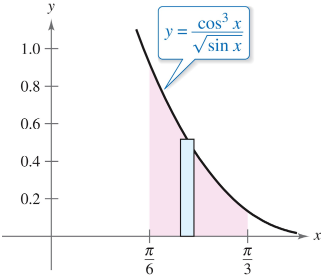


Figure 4: The area of the region is approximately 0.239.

## Example 3 (Power of cosine is even and nonnegative)

Find $\int \cos^4 x \, \mathrm{d}x$.

$$\begin{aligned}
\int \cos^4 x \, \mathrm{d}x &= \int \left(\frac{1+\cos 2x}{2}\right)^2 \mathrm{d}x = \int \left(\frac{1}{4} + \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4}\right) \mathrm{d}x \\
&= \int \left[\frac{1}{4} + \frac{\cos 2x}{2} + \frac{1}{4}\left(\frac{1+\cos 4x}{2}\right)\right] \mathrm{d}x \\
&= \frac{3}{8}\int \mathrm{d}x + \frac{1}{4}\int 2\cos 2x \, \mathrm{d}x + \frac{1}{32}\int 4\cos 4x \, \mathrm{d}x \\
&= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C
\end{aligned}$$

■

Wallis's Formulas

**a.** If $n$ is odd $(n \geq 3)$, then

$$\int_0^{\pi/2} \cos^n x \, \mathrm{d}x = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right).$$

**b.** If $n$ is even $(n \geq 2)$, then

$$\int_0^{\pi/2} \cos^n x \, \mathrm{d}x = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right).$$

These formulas are also valid if $\cos^n x$ is replaced by $\sin^n x$.

# Integrals involving powers of secant and tangent

- The following guidelines can help you evaluate integrals of the form $\int \sec^m x \tan^n x \, \mathrm{d}x$

**Guidelines for evaluating integrals involving powers of secant and tangent**

1. If the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then, expand and integrate.

$$\int \sec^{\overbrace{2k}^{\text{even}}} x \tan^n x \, \mathrm{d}x = \int \overbrace{(\sec^2 x)^{k-1}}^{\text{Convert to tangents}} \tan^n x \overbrace{\sec^2 x \, \mathrm{d}x}^{\text{Save for } \mathrm{d}u}$$

$$= \int (1 + \tan^2 x)^{k-1} \tan^n x \sec^2 x \, \mathrm{d}x$$

2. If the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then, expand and integrate.

$$\int \sec^m x \tan^{\overbrace{2k+1}^{\text{Odd}}} x \, dx = \int \sec^{m-1} x \quad \overbrace{(\tan^2 x)^k}^{\text{Convert to secants}} \quad \overbrace{\sec x \tan x \, dx}^{\text{Save for } du}$$

$$= \int \sec^{m-1} x (\sec^2 x - 1)^k \sec x \tan x \, dx$$

3. If there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

$$\int \tan^n x \, dx = \int \tan^{n-2} x \quad \overbrace{(\tan^2 x)}^{\text{Convert to secants}} \quad dx$$

$$= \int \tan^{n-2} x (\sec^2 x - 1) \, dx$$

4. If the integral is of the form $\int \sec^m x \, \mathrm{d}x$, where $m$ is odd and positive, use integration by parts, as illustrated in Example 5 in the preceding section.
5. If none of the first four guidelines applies, try converting to sines and cosines.

**Example 4 (Power of tangent is odd and positive)**

Find $\int \frac{\tan^3 x}{\sqrt{\sec x}} \, \mathrm{d}x$.

- Because you expect to use the Power Rule with $u = \sec x$, save a factor of $(\sec x \tan x)$ to form $\mathrm{d}u$ and convert the remaining tangent factors to secants.

$$
\begin{aligned}
\int \frac{\tan^3 x}{\sqrt{\sec x}}\,\mathrm{d}x &= \int (\sec x)^{-1/2} \tan^3 x \,\mathrm{d}x \\
&= \int (\sec x)^{-3/2} (\tan^2 x)(\sec x \tan x)\,\mathrm{d}x \\
&= \int (\sec x)^{-3/2} (\sec^2 x - 1)(\sec x \tan x)\,\mathrm{d}x \\
&= \int [(\sec x)^{1/2} - (\sec x)^{-3/2}](\sec x \tan x)\,\mathrm{d}x \\
&= \frac{2}{3}(\sec x)^{3/2} + 2(\sec x)^{-1/2} + C \qquad \blacksquare
\end{aligned}
$$

## Example 5 (Power of secant is even and positive)

Find $\int \sec^4 3x \tan^3 3x \, \mathrm{d}x$.

Let $u = \tan 3x$. Then $\mathrm{d}u = 3\sec^2 3x \, \mathrm{d}x$ and you can write

$$\begin{aligned}
\int \sec^4 3x \tan^3 3x \, \mathrm{d}x &= \int \sec^2 3x \tan^3 3x (\sec^2 3x) \, \mathrm{d}x \\
&= \int (1 + \tan^2 3x) \tan^3 3x (\sec^2 3x) \, \mathrm{d}x \\
&= \frac{1}{3} \int (\tan^3 3x + \tan^5 3x)(3\sec^2 3x) \, \mathrm{d}x \\
&= \frac{1}{3} \left( \frac{\tan^4 3x}{4} + \frac{\tan^6 3x}{6} \right) + C \\
&= \frac{\tan^4 3x}{12} + \frac{\tan^6 3x}{18} + C.
\end{aligned}$$

■

## Example 6 (Power of tangent is even)

Evaluate $\int_0^{\pi/4} \tan^4 x \, \mathrm{d}x$.

- Because there are no secant factors, you can begin by converting a tangent squared factor to a secant-squared factor.

$$\begin{aligned}
\int \tan^4 x \, \mathrm{d}x &= \int \tan^2 x (\tan^2 x) \, \mathrm{d}x = \int \tan^2 x (\sec^2 x - 1) \, \mathrm{d}x \\
&= \int \tan^2 x \sec^2 x \, \mathrm{d}x - \int \tan^2 x \, \mathrm{d}x \\
&= \int \tan^2 x \sec^2 x \, \mathrm{d}x - \int (\sec^2 x - 1) \, \mathrm{d}x \\
&= \frac{\tan^3 x}{3} - \tan x + x + C
\end{aligned}$$

- You can evaluate the definite integral as follows.

$$\int_0^{\pi/4} \tan^4 x \, \mathrm{d}x = \left[ \frac{\tan^3 x}{3} - \tan x + x \right]_0^{\pi/4} = \frac{\pi}{4} - \frac{2}{3} \approx 0.119$$

- The area represented by the definite integral is shown in Figure 5.
- Try using Midpoint's Rule to approximate this integral. With $n = 15$, you should obtain an approximation that is within 0.001 of the actual value. ■

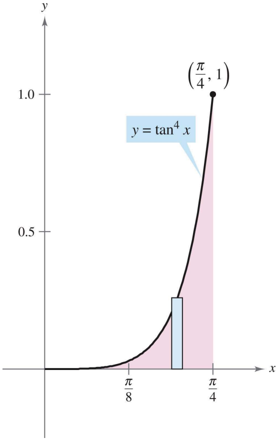


Figure 5: The area of the region is approximately 0.119.

## Example 7 (Converting to sines and cosines)

Find $\int \frac{\sec x}{\tan^2 x}\,\mathrm{d}x$.

$$\int \frac{\sec x}{\tan^2 x}\,\mathrm{d}x = \int \left(\frac{1}{\cos x}\right)\left(\frac{\cos x}{\sin x}\right)^2 \mathrm{d}x = \int (\sin x)^{-2}\cos x\,\mathrm{d}x$$
$$= -(\sin x)^{-1} + C = -\csc x + C \quad \blacksquare$$

# Integrals involving sine-cosine products with different angles

- Integrals involving the products of sines and cosines of two different angles occur in many applications.
- In such instances you can use the following product-to-sum identities.

$$\begin{aligned}
\sin mx \sin nx &= \frac{1}{2}(\cos[(m-n)x] - \cos[(m+n)x]) \\
\sin mx \cos nx &= \frac{1}{2}(\sin[(m-n)x] + \sin[(m+n)x]) \\
\cos mx \cos nx &= \frac{1}{2}(\cos[(m-n)x] + \cos[(m+n)x])
\end{aligned}$$

## Example 8 (Using Product-to-Sum Identities)

Find $\int \sin 5x \cos 4x \,\mathrm{d}x$.

Considering the second product-to-sum identity above, you can write

$$\int \sin 5x \cos 4x \,\mathrm{d}x = \frac{1}{2}\int (\sin x + \sin 9x)\,\mathrm{d}x = \frac{1}{2}\left(-\cos x - \frac{\cos 9x}{9}\right) + C$$
$$= -\frac{\cos x}{2} - \frac{\cos 9x}{18} + C. \qquad \blacksquare$$

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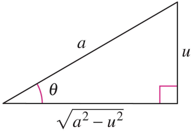
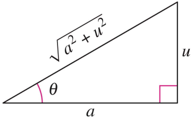
# Trigonometric substitution

- Use trigonometric substitution to evaluate integrals involving the radicals
$$\sqrt{a^2 - u^2}, \quad \sqrt{a^2 + u^2}, \quad \text{and} \quad \sqrt{u^2 - a^2}.$$
- The objective with trigonometric substitution is to eliminate the radical in the integrand. You do this by using the Pythagorean identities
$$\cos^2\theta = 1 - \sin^2\theta, \quad \sec^2\theta = 1 + \tan^2\theta, \quad \text{and} \quad \tan^2\theta = \sec^2\theta - 1$$
- For example, if $a > 0$, let $u = a\sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$. Then
$$\sqrt{a^2 - u^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta.$$
- Note that $\cos\theta \geq 0$, because $-\pi/2 \leq \theta \leq \pi/2$.

1. For integrals involving $\sqrt{a^2 - u^2}$, let $u = a\sin\theta$. Then $\sqrt{a^2 - u^2} = a\cos\theta$, where $-\pi/2 \le \theta \le \pi/2$.

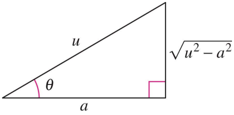



2. For integrals involving $\sqrt{a^2 + u^2}$, let $u = a\tan\theta$. Then $\sqrt{a^2 + u^2} = a\sec\theta$, where $-\pi/2 \le \theta \le \pi/2$.




3. For integrals involving $\sqrt{u^2 - a^2}$, let $u = a\sec\theta$.

Then $\sqrt{u^2 - a^2} = \begin{cases} a\tan\theta & \text{if } u > a, \text{ where } 0 \le \theta < \pi/2 \\ -a\tan\theta, & \text{if } u < -a, \text{ where } \pi/2 < \theta \le \pi. \end{cases}$



## Example 1 (Trigonometric substitution: $u = a\sin\theta$)

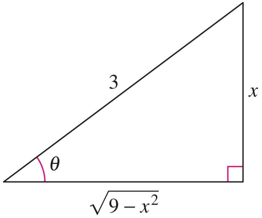
Find $\int \frac{\mathrm{d}x}{x^2\sqrt{9-x^2}}$.

- First, note that none of the basic integration rules applies.
- To use trigonometric substitution, you should observe that $\sqrt{9-x^2}$ is of the form $\sqrt{a^2-u^2}$.
- So, you can use the substitution

$$x = a\sin\theta = 3\sin\theta.$$

- Using differentiation and the triangle shown below, you obtain

$$\mathrm{d}x = 3\cos\theta\,\mathrm{d}\theta, \quad \sqrt{9-x^2} = 3\cos\theta, \quad \text{and} \quad x^2 = 9\sin^2\theta.$$



- So, trigonometric substitution yields

$$\int \frac{\mathrm{d}x}{x^2\sqrt{9-x^2}} = \int \frac{3\cos\theta\,\mathrm{d}\theta}{(9\sin^2\theta)(3\cos\theta)} = \frac{1}{9}\int \frac{\mathrm{d}\theta}{\sin^2\theta} = \frac{1}{9}\int \csc^2\theta\,\mathrm{d}\theta$$
$$= -\frac{1}{9}\cot\theta + C$$
$$= -\frac{1}{9}\left(\frac{\sqrt{9-x^2}}{x}\right) + C = -\frac{\sqrt{9-x^2}}{9x} + C.$$

- Note that the triangle in the Figure can be used to convert the $\theta$'s back to $x$'s, as follows.

$$\cot\theta = \frac{\text{adj.}}{\text{opp.}} = \frac{\sqrt{9-x^2}}{x}$$

■

## Example 2 (Trigonometric substitution: $u = a\tan\theta$)

Find $\int \frac{\mathrm{d}x}{\sqrt{4x^2+1}}$.

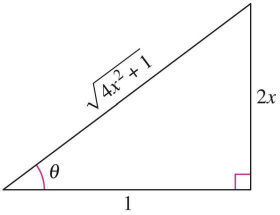
- Let $u = 2x$, $a = 1$, and $2x = \tan\theta$, as shown below. Then,

$$\mathrm{d}x = \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta \quad \text{and} \quad \sqrt{4x^2+1} = \sec\theta.$$

- Trigonometric substitution produces

$$\begin{aligned}\int \frac{1}{\sqrt{4x^2+1}}\,\mathrm{d}x &= \frac{1}{2}\int \frac{\sec^2\theta}{\sec\theta}\,\mathrm{d}\theta = \frac{1}{2}\int \sec\theta\,\mathrm{d}\theta \\ &= \frac{1}{2}\ln|\sec\theta + \tan\theta| + C \\ &= \frac{1}{2}\ln\left|\sqrt{4x^2+1} + 2x\right| + C.\end{aligned}$$

■




## Example 3 (Trigonometric substitution: rational powers)

Find $\int \frac{\mathrm{d}x}{(x^2+1)^{3/2}}$.

Let $x = \tan\theta$, then $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$ and $x^2+1 = \tan^2\theta + 1 = \sec^2\theta$.

$$\begin{aligned}\int \frac{1}{(x^2+1)^{3/2}}\,\mathrm{d}x &= \int \frac{\sec^2\theta}{\sec^3\theta}\,\mathrm{d}\theta = \int \frac{\mathrm{d}\theta}{\sec\theta} \\ &= \int \cos\theta\,\mathrm{d}\theta = \sin\theta + C \\ &= \frac{x}{\sqrt{x^2+1}} + C\end{aligned}$$

■

## Example 4 (Converting the limits of integration)

Evaluate $\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,\mathrm{d}x$.

- Because $\sqrt{x^2-3}$ has the form $\sqrt{u^2-a^2}$, you can consider

$$u = x, \qquad a = \sqrt{3}, \quad \text{and} \quad x = \sqrt{3}\sec\theta$$

  as shown in Figure 6.

- Then,

$$\mathrm{d}x = \sqrt{3}\sec\theta\tan\theta\,\mathrm{d}\theta \quad \text{and} \quad \sqrt{x^2-3} = \sqrt{3}\tan\theta.$$

- To determine the upper and lower limits of integration, use the substitution $x = \sqrt{3}\sec\theta$, as follows.

$$\begin{array}{lllll} \text{Lower Limit:} & \text{When } x = \sqrt{3} & \Longrightarrow & \sec\theta = 1 & \Longrightarrow & \theta = 0. \\ \text{Upper Limit:} & \text{When } x = 2 & \Longrightarrow & \sec\theta = \frac{2}{\sqrt{3}} & \Longrightarrow & \theta = \frac{\pi}{6}. \end{array}$$

- So, you have

$$\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,\mathrm{d}x = \int_0^{\pi/6} \frac{(\sqrt{3}\tan\theta)(\sqrt{3}\sec\theta\tan\theta)}{\sqrt{3}\sec\theta}\,\mathrm{d}\theta$$

$$= \int_0^{\pi/6} \sqrt{3}\tan^2\theta\,\mathrm{d}\theta$$

$$= \sqrt{3}\int_0^{\pi/6} (\sec^2\theta - 1)\,\mathrm{d}\theta = \sqrt{3}\,[\tan\theta - \theta]_0^{\pi/6}$$

$$= \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right) = 1 - \frac{\sqrt{3}\pi}{6} \approx 0.0931. \qquad \blacksquare$$

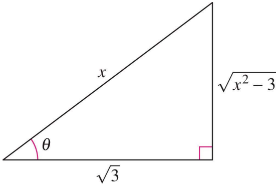



Figure 6: $\sec\theta = \frac{x}{\sqrt{3}}$, $\tan\theta = \frac{\sqrt{x^2-3}}{\sqrt{3}}$.

## Theorem 8.2 (Special integration formulas ($a > 0$))

1. $\int \sqrt{a^2 - u^2}\,\mathrm{d}u = \frac{1}{2}\left(a^2 \arcsin \frac{u}{a} + u\sqrt{a^2 - u^2}\right) + C$
2. $\int \sqrt{u^2 - a^2}\,\mathrm{d}u = \frac{1}{2}\left(u\sqrt{u^2 - a^2} - a^2 \ln\left|u + \sqrt{u^2 - a^2}\right|\right) + C, \quad u > a$
3. $\int \sqrt{u^2 + a^2}\,\mathrm{d}u = \frac{1}{2}\left(u\sqrt{u^2 + a^2} + a^2 \ln\left|u + \sqrt{u^2 + a^2}\right|\right) + C$

**Example 5 (Finding arc length)**

Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$ (see Figure 7).

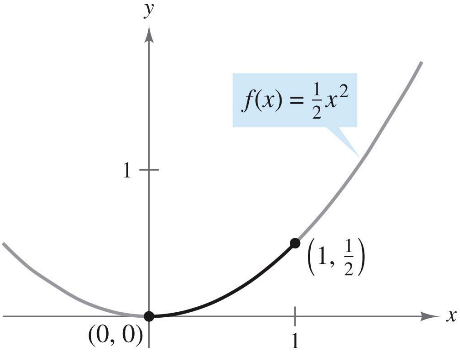



Figure 7: The arc length of the curve of $f(x) = \frac{1}{2}x^2$.

Refer to the arc length formula and let $x = \tan\theta$. Then

$$
\begin{aligned}
s &= \int_0^1 \sqrt{1 + [f'(x)]^2}\,\mathrm{d}x = \int_0^1 \sqrt{1 + x^2}\,\mathrm{d}x = \int_0^{\pi/4} \sec^3\theta\,\mathrm{d}\theta \\
&= \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^{\pi/4} \\
&= \frac{1}{2}\left[\sqrt{2} + \ln(\sqrt{2} + 1)\right] \approx 1.148.
\end{aligned}
$$

■

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# Partial fractions

- The Method of Partial Fractions is a procedure for decomposing a rational function into simpler rational functions to which you can apply the basic integration formulas.
- To see the benefit of the Method of Partial Fractions, consider the integral

$$\int \frac{1}{x^2 - 5x + 6}\,\mathrm{d}x.$$

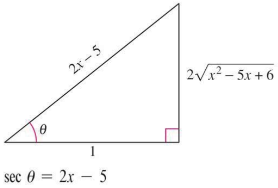


Figure 8: Trigonometric substitution.

- To evaluate this integral without partial fractions, you can complete the square and use trigonometric substitution (see Figure 8) to obtain

$$
\begin{aligned}
\int \frac{1}{x^2-5x+6}\,\mathrm{d}x &= \int \frac{\mathrm{d}x}{(x-5/2)^2-(1/2)^2} \qquad a=\frac{1}{2},\; x-\frac{5}{2}=\frac{1}{2}\sec\theta \\
&= \int \frac{(1/2)\sec\theta\tan\theta\,\mathrm{d}\theta}{(1/4)\tan^2\theta} \qquad \mathrm{d}x=\frac{1}{2}\sec\theta\tan\theta\,\mathrm{d}\theta \\
&= 2\int \csc\theta\,\mathrm{d}\theta = -2\ln|\csc\theta+\cot\theta|+C \\
&= 2\ln|\csc\theta-\cot\theta|+C \\
&= 2\ln\left|\frac{2x-5}{2\sqrt{x^2-5x+6}}-\frac{1}{2\sqrt{x^2-5x+6}}\right|+C \\
&= 2\ln\left|\frac{x-3}{\sqrt{x^2-5x+6}}\right|+C \\
&= 2\ln\left|\frac{\sqrt{x-3}}{\sqrt{x-2}}\right|+C = \ln\left|\frac{x-3}{x-2}\right|+C \\
&= \ln|x-3|-\ln|x-2|+C.
\end{aligned}
$$

- Now, suppose you had observed that

$$\frac{1}{x^2-5x+6} = \frac{1}{x-3} - \frac{1}{x-2}. \qquad \text{Partial fraction decomposition}$$

- Then you could evaluate the integral easily, as follows.

$$\begin{aligned}\int \frac{1}{x^2-5x+6}\,\mathrm{d}x &= \int \left(\frac{1}{x-3} - \frac{1}{x-2}\right)\mathrm{d}x \\ &= \ln|x-3| - \ln|x-2| + C\end{aligned}$$

- This method is clearly preferable to trigonometric substitution. However, its use depends on the ability to factor the denominator, $x^2-5x+6$, and to find the partial fractions

$$\frac{1}{x-3} \quad \text{and} \quad -\frac{1}{x-2}.$$

1. Divide if improper: If $N(x)/D(x)$ is an improper fraction (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

2. Factor denominator: Completely factor the denominator into factors of the form

$$(px+q)^m \quad \text{and} \quad (ax^2+bx+c)^n$$

where $ax^2+bx+c$ is irreducible.

3. Linear factors: For each factor of the form $(px+q)^m$, the partial fraction decomposition must include the following sum of $m$ fractions.

$$\frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}$$

4. Quadratic factors: For each factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition must include the following sum of $n$ fractions.

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

**Example 1 (Distinct linear factors)**

Write the partial fraction decomposition for $\frac{1}{x^2-5x+6}$.

- Because $x^2 - 5x + 6 = (x-3)(x-2)$, you should include one partial fraction for each factor and write

$$\frac{1}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2}$$

where $A$ and $B$ are to be determined.

- Multiplying this equation by the least common denominator $(x-3)(x-2)$ yields the

$$1 = A(x-2) + B(x-3). \qquad \text{Basic equation}$$

- Because this equation is to be true for all $x$, you can substitute any convenient values for $x$ to obtain equations in $A$ and $B$. The most convenient values are the ones that make particular factors equal to 0.

- To solve for $A$, let $x = 3$ and obtain

$$1 = A(3-2) + B(3-3) \qquad 1 = A(1) + B(0) \qquad A = 1.$$

- To solve for $B$, let $x = 2$ and obtain

$$1 = A(2-2) + B(2-3) \qquad 1 = A(0) + B(-1) \qquad B = -1.$$

- So, the decomposition is

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{x-3} - \frac{1}{x-2}$$

as shown at the beginning of this section. ∎

## Example 2 (Repeated linear factors)

Find $\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,\mathrm{d}x$.

- Because
$$x^3 + 2x^2 + x = x(x^2 + 2x + 1) = x(x+1)^2$$
you should include one fraction for each power of $x$ and $(x+1)$ and write
$$\frac{5x^2 + 20x + 6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$
- Multiplying by the least common denominator $x(x+1)^2$ yields the basic equation
$$5x^2 + 20x + 6 = A(x+1)^2 + Bx(x+1) + Cx.$$
- To solve for $A$, let $x = 0$. This eliminates the $B$ and $C$ terms and yields
$$6 = A(1) + 0 + 0 \qquad\qquad A = 6.$$

- To solve for $C$, let $x = -1$. This eliminates the $A$ and $B$ terms and yields

$$5 - 20 + 6 = 0 + 0 - C \qquad\qquad C = 9.$$

- The most convenient choices for $x$ have been used, so to find the value of $B$, you can use any other value of $x$ along with the calculated values of $A$ and $C$.
- Using $x = 1$, $A = 6$, and $C = 9$ produces

$$\begin{aligned} 5 + 20 + 6 &= A(4) + B(2) + C \\ 31 &= 6(4) + 2B + 9 \\ -2 &= 2B \qquad B = -1. \end{aligned}$$

- So, it follows that

$$\int \frac{5x^2+20x+6}{x(x+1)^2}\,\mathrm{d}x = \int \left(\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\right)\,\mathrm{d}x$$
$$= 6\ln|x| - \ln|x+1| + 9\frac{(x+1)^{-1}}{-1} + C$$
$$= \ln\left|\frac{x^6}{x+1}\right| - \frac{9}{x+1} + C.$$

- Try checking this result by differentiating. Include algebra in your check, simplifying the derivative until you have obtained the original integrand. ■

# Quadratic factors

## Example 3 (Distinct linear and quadratic factors)

Find $\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\,\mathrm{d}x$.

- Because $(x^2-x)(x^2+4) = x(x-1)(x^2+4)$ you should include one partial fraction for each factor and write

$$\frac{2x^3-4x-8}{x(x-1)(x^2+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}$$

- Multiplying by the least common denominator $x(x-1)(x^2+4)$ yields the basic equation
$2x^3-4x-8 = A(x-1)(x^2+4) + Bx(x^2+4) + (Cx+D)(x)(x-1)$.
- To solve for $A$, let $x=0$ and obtain

$$-8 = A(-1)(4) + 0 + 0 \quad \Longrightarrow \quad 2 = A.$$

- To solve for $B$, let $x = 1$ and obtain

$$-10 = 0 + B(5) + 0 \quad \Longrightarrow \quad -2 = B.$$

- At this point, $C$ and $D$ are yet to be determined.
- You can find these remaining constants by choosing two other values for $x$ and solving the resulting system of linear equations.
- If $x = -1$, then, using $A = 2$ and $B = -2$, you can write

$$\begin{aligned} -6 &= (2)(-2)(5) + (-2)(-1)(5) + (-C + D)(-1)(-2) \\ 2 &= -C + D. \end{aligned}$$

- If $x = 2$, you have

$$\begin{aligned} 0 &= (2)(1)(8) + (-2)(2)(8) + (2C + D)(2)(1) \\ 8 &= 2C + D. \end{aligned}$$

- Solving the linear system by subtracting the first equation from the second

$$\begin{aligned} -C + D &= 2 \\ 2C + D &= 8 \end{aligned}$$

yields $C = 2$. Consequently, $D = 4$, and it follows that

$$\begin{aligned} &\int \frac{2x^3 - 4x - 8}{x(x-1)(x^2+4)} \, \mathrm{d}x \\ = &\int \left( \frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2+4} + \frac{4}{x^2+4} \right) \mathrm{d}x \\ = &\, 2\ln|x| - 2\ln|x-1| + \ln(x^2+4) + 2\arctan\frac{x}{2} + C. \end{aligned}$$

■

## Example 4 (Repeated quadratic factors)

Find $\int \frac{8x^3+13x}{(x^2+2)^2}\,\mathrm{d}x$.

- Include one partial fraction for each power of $(x^2+2)$ and write

$$\frac{8x^3+13x}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}.$$

- Multiplying by the least common denominator $(x^2+2)^2$ yields the basic equation

$$8x^3+13x = (Ax+B)(x^2+2) + Cx + D.$$

- Expanding the basic equation and collecting like terms produces

$$8x^3+13x = Ax^3+2Ax+Bx^2+2B+Cx+D$$
$$8x^3+13x = Ax^3+Bx^2+(2A+C)x+(2B+D).$$

- Now, you can equate the coefficients of like terms on opposite sides of the equation.

- Using the known values $A = 8$ and $B = 0$, you can write

$$\begin{aligned} 13 = 2A + C = 2(8) + C &\implies C = -3 \\ 0 = 2B + D = 2(0) + D &\implies D = 0. \end{aligned}$$

- Finally, you can conclude that

$$\begin{aligned} \int \frac{8x^3 + 13x}{(x^2+2)^2}\,\mathrm{d}x &= \int \left( \frac{8x}{x^2+2} + \frac{-3x}{(x^2+2)^2} \right) \mathrm{d}x \\ &= 4\ln(x^2+2) + \frac{3}{2(x^2+2)} + C. \end{aligned}$$

■

Guidelines for solving the basic equation

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Linear Factors

1. Substitute the roots of the distinct linear factors in the basic equation.
2. For repeated linear factors, use the coefficients determined in guideline 1 to rewrite the basic equation. Then substitute other convenient values of $x$ and solve for the remaining coefficients.

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Quadratic Factors

1. Expand the basic equation.
2. Collect terms according to powers of $x$.
3. Equate the coefficients of like powers to obtain a system of linear equations involving $A$, $B$, $C$, and so on.
4. Solve the system of linear equations.

1. It is not necessary to use the partial fractions technique on all rational functions.

$$\int \frac{x^2+1}{x^3+3x-4}\,\mathrm{d}x = \frac{1}{3}\int \frac{3x^2+3}{x^3+3x-4}\,\mathrm{d}x = \frac{1}{3}\ln\left|x^3+3x-4\right| + C$$

2. If the integrand is not in reduced form, reducing it may eliminate the need for partial fractions.

$$\begin{aligned}\int \frac{x^2-x-2}{x^3-2x-4}\,\mathrm{d}x &= \int \frac{(x+1)(x-2)}{(x-2)(x^2+2x+2)}\,\mathrm{d}x \\ &= \int \frac{x+1}{x^2+2x+2}\,\mathrm{d}x = \frac{1}{2}\ln\left|x^2+2x+2\right| + C\end{aligned}$$

3. Finally, partial fractions can be used with some quotients involving transcendental functions. For instance, the substitution $u = \sin x$ allows you to write

$$\int \frac{\cos x}{\sin x(\sin x - 1)}\,\mathrm{d}x = \int \frac{\mathrm{d}u}{u(u-1)}. \qquad u = \sin x,\ \mathrm{d}u = \cos x\,\mathrm{d}x$$

# Table of Contents

# Improper integrals with infinite limits of integration

- The definition of a definite integral

$$\int_a^b f(x)\,\mathrm{d}x$$

requires that the interval $[a, b]$ be finite.

- A procedure for evaluating integrals that do not satisfy these requirements-usually because either one or both of the limits of integration are infinite, or $f$ has a finite number of infinite discontinuities in the interval $[a, b]$.
- Integrals that possess either property are improper integrals.
- A function $f$ is said to have an infinite discontinuity at $c$ if, from the right or left,

$$\lim_{x\to c} f(x) = \infty \qquad \text{or} \qquad \lim_{x\to c} f(x) = -\infty.$$

## Definition 8.1 (Improper integrals with infinite integration limits)

1. If $f$ is continuous on the interval $[a, \infty)$, then

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x.$$

2. If $f$ is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{a\to-\infty} \int_a^b f(x)\,\mathrm{d}x.$$

3. If $f$ is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^c f(x)\,\mathrm{d}x + \int_c^\infty f(x)\,\mathrm{d}x$$

where $c$ is any real number.

In the above first two cases, the improper integral converges if the limit exists-otherwise, the improper integral diverges. In the third case, the improper integral on the left diverges if either of the improper integral on the right diverges.

**Example 1 (An improper integral that diverges)**

Evaluate $\int_1^\infty \frac{\mathrm{d}x}{x}$.

$$\int_1^\infty \frac{\mathrm{d}x}{x} = \lim_{b\to\infty} \int_1^b \frac{\mathrm{d}x}{x} = \lim_{b\to\infty} [\ln x]_1^b = \lim_{b\to\infty} (\ln b - 0) = \infty$$

See Figure 9. ■

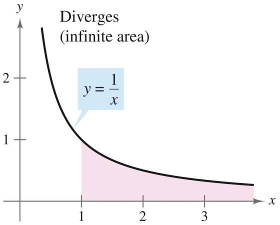



Figure 9: The unbounded region has infinite area.

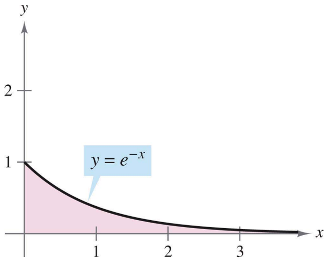
**Example 2 (Improper integrals that converge)**

Evaluate each improper integral.

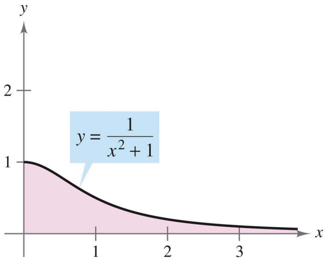
**a.** $\int_0^{\infty} e^{-x}\,\mathrm{d}x$ **b.** $\int_0^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x$

$$\textbf{a.}\ \int_0^\infty e^{-x}\,\mathrm{d}x = \lim_{b\to\infty}\int_0^b e^{-x}\,\mathrm{d}x = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = \lim_{b\to\infty}\left(-e^{-b}+1\right) = 1$$

$$\textbf{b.}\ \int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,\mathrm{d}x = \lim_{b\to\infty}\left[\arctan x\right]_0^b = \lim_{b\to\infty}\arctan b = \frac{\pi}{2} \quad \blacksquare$$




(a) The area of the unbounded region is 1.




(b) The area of the unbounded region is $\pi/2$.

## Example 3 (Using L'Hôpital's Rule with an improper integral)

Evaluate $\int_1^{\infty}(1-x)e^{-x}\,\mathrm{d}x$.

- Use integration by parts, with $\mathrm{d}v = e^{-x}\,\mathrm{d}x$ and $u = (1-x)$.

$$\begin{aligned}\int (1-x)e^{-x}\,\mathrm{d}x &= -e^{-x}(1-x) - \int e^{-x}\,\mathrm{d}x \\ &= -e^{-x} + xe^{-x} + e^{-x} + C = xe^{-x} + C\end{aligned}$$

- Now, apply the definition of an improper integral.

$$\int_1^{\infty}(1-x)e^{-x}\,\mathrm{d}x = \lim_{b\to\infty}\left[xe^{-x}\right]_1^b = \left(\lim_{b\to\infty}\frac{b}{e^b}\right) - \frac{1}{e}$$

- Finally, using L'Hôpital's Rule on the right-hand limit produces

$$\lim_{b\to\infty}\frac{b}{e^b} = \lim_{b\to\infty}\frac{1}{e^b} = 0$$

from which you can conclude that

$$\int_1^{\infty}(1-x)e^{-x}\,\mathrm{d}x = -\frac{1}{e}.$$

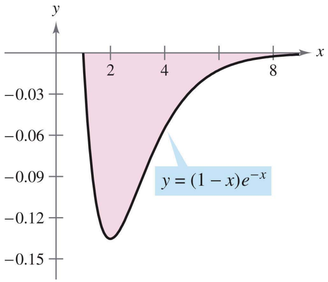



Figure 11: The area of the unbounded region is $1/e$.

## Example 4 (Infinite upper and lower limits of integration)

Evaluate $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,\mathrm{d}x$.

- Note that the integrand is continuous on $(-\infty, \infty)$.

- To evaluate the integral, you can break it into two parts, choosing $c = 0$ as a convenient value.

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\,dx + \int_{0}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$$
$$= \lim_{b\to-\infty} \left[\tan^{-1} e^x\right]_b^0 + \lim_{b\to\infty} \left[\tan^{-1} e^x\right]_0^b$$
$$= \lim_{b\to-\infty} \left(\frac{\pi}{4} - \tan^{-1} e^b\right) + \lim_{b\to\infty} \left(\tan^{-1} e^b - \frac{\pi}{4}\right)$$
$$= \frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2}$$

See Figure 12. ■

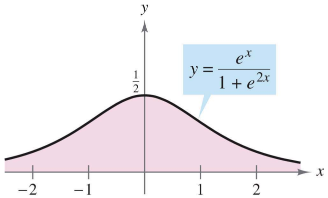



Figure 12: The area of the unbounded region is $\pi/2$.

## Definition 8.2 (Improper integrals with infinite discontinuities)

1. If $f$ is continuous on the interval $[a, b)$ and has an infinite discontinuity at $b$, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{c \to b^-} \int_a^c f(x)\,\mathrm{d}x.$$

2. If $f$ is continuous on the interval $(a, b]$ and has an infinite discontinuity at $a$, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{c \to a^+} \int_c^b f(x)\,\mathrm{d}x.$$

3. If $f$ is continuous on the interval $[a, b]$, except for some $c$ in $(a, b)$ at which $f$ has an infinite discontinuity, then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

In the above first two cases, the improper integral converges if the limit exists-otherwise, the improper integral diverges. In the third case, the improper integral on the left diverges if either of the improper integral on the right diverges.

**Example 6 (An improper integral with an infinite discontinuity)**

Evaluate $\int_0^1 \frac{\mathrm{d}x}{\sqrt[3]{x}}$.

- The integrand has an infinite discontinuity at $x = 0$, as shown in Figure 13.
- You can evaluate this integral as shown below.

$$\int_0^1 x^{-1/3}\,\mathrm{d}x = \lim_{b\to 0^+}\left[\frac{x^{2/3}}{2/3}\right]_b^1 = \lim_{b\to 0^+}\frac{3}{2}(1 - b^{2/3}) = \frac{3}{2} \qquad \blacksquare$$

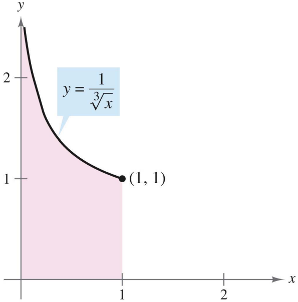



Figure 13: Infinite discontinuity at $x = 0$.

## Example 7 (An improper integrals that diverges)

Evaluate $\int_0^2 \frac{\mathrm{d}x}{x^3}$.

$$\int_0^2 \frac{\mathrm{d}x}{x^3} = \lim_{b \to 0^+} \left[ -\frac{1}{2x^2} \right]_b^2 = \lim_{b \to 0^+} \left( -\frac{1}{8} + \frac{1}{2b^2} \right) = \infty \qquad \blacksquare$$

### Example 8 (An improper integrals with an interior discontinuity)

Evaluate $\int_{-1}^{2} \frac{\mathrm{d}x}{x^3}$.

$$\int_{-1}^{2} \frac{\mathrm{d}x}{x^3} = \int_{-1}^{0} \frac{\mathrm{d}x}{x^3} + \int_{0}^{2} \frac{\mathrm{d}x}{x^3}$$

From Example 7 you know that the second integral diverges. So, the original improper integral also diverges. ■

### Example 9 (A doubly improper integral)

Evaluate $\int_{0}^{\infty} \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x$.

- To evaluate this integral, split it at a convenient point (say, $x = 1$) and write

$$\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x = \int_0^1 \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x + \int_1^\infty \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x$$
$$= \lim_{b\to 0^+}\left[2\tan^{-1}\sqrt{x}\right]_b^1 + \lim_{c\to\infty}\left[2\tan^{-1}\sqrt{x}\right]_1^c$$
$$= 2\left(\frac{\pi}{4}\right) - 0 + 2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{4}\right) = \pi.$$

■

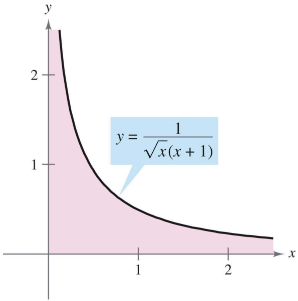



Figure 14: The area of the unbounded region is $\pi$.

## Example 10 (An application involving arc length)

Use the formula for arc length to show that the circumference of the circle $x^2 + y^2 = 1$ is $2\pi$.

- To simplify the work, consider the quarter circle given by $y = \sqrt{1 - x^2}$, where $0 \leq x \leq 1$.
- The function $y$ is differentiable for any $x$ in this interval except $x = 1$.
- Therefore, the arc length of the quarter circle is given by the improper integral

$$s = \int_0^1 \sqrt{1 + (y')^2}\,\mathrm{d}x = \int_0^1 \sqrt{1 + \left(\frac{-x}{\sqrt{1 - x^2}}\right)^2}\,\mathrm{d}x$$
$$= \int_0^1 \frac{1}{\sqrt{1 - x^2}}\,\mathrm{d}x.$$

- This integral is improper because it has an infinite discontinuity at $x = 1$. So, you can write

$$s = \int_0^1 \frac{1}{\sqrt{1 - x^2}}\, \mathrm{d}x = \lim_{b \to 1^-} \left[\sin^{-1} x\right]_0^b = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

- Finally, multiplying by 4, you can conclude that the circumference of the circle is $4s = 2\pi$, as shown in Figure 15. ■

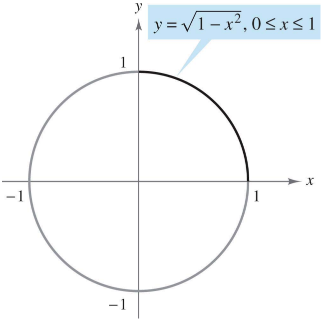



Figure 15: The circumference of the circle is $2\pi$.

## Theorem 8.7 (A special type of improper integral)

$$\int_1^\infty \frac{\mathrm{d}x}{x^p} = \begin{cases} \frac{1}{p-1}, & \textit{if } p > 1 \\ \textit{diverges}, & \textit{if } p \le 1 \end{cases}$$

If $p = 1$,

$$\int_1^\infty \frac{1}{x}\,\mathrm{d}x = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,\mathrm{d}x = \lim_{b\to\infty} [\ln x]_1^b = \lim_{b\to\infty} \ln b = \infty.$$

Diverges. For $p \neq 1$,

$$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \lim_{b\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^b = \lim_{b\to\infty} \left(\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right).$$

This converges to $\frac{1}{p-1}$, if $1 - p < 0$ or $p > 1$. $\square$

## Example 11 (An application involving a solid of revolution)

The solid formed by revolving (about the $x$-axis) the unbounded region lying between the graph of $f(x) = 1/x$ and the $x$-axis ($x \geq 1$) is called Gabriel's Horn. (See Figure 16.) Show that this solid has a finite volume and an infinite surface area.

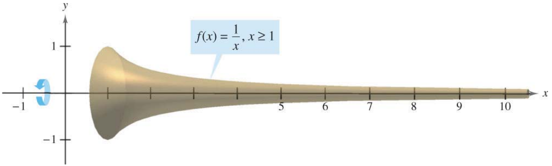



Figure 16: Gabriel's Horn has a finite volume and an infinite surface area.

- Using the Disk Method and Theorem 8.7, you can determine the volume to be

$$V = \pi \int_1^{\infty} \left(\frac{1}{x}\right)^2 \mathrm{d}x = \pi \left(\frac{1}{2-1}\right) = \pi.$$

- The surface area is given by

$$S = 2\pi \int_1^\infty f(x)\sqrt{1 + [f'(x)]^2}\,\mathrm{d}x = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,\mathrm{d}x.$$

- Because

$$\sqrt{1 + \frac{1}{x^4}} > 1$$

on the interval $[1, \infty)$, and the improper integral

$$\int_1^\infty \frac{1}{x}\,\mathrm{d}x$$

diverges, you can conclude that the improper integral

$$\int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,\mathrm{d}x$$

also diverges. So, the surface area is infinite. ■

- In some cases, it is impossible or hard to find the exact value of an improper integral, but it is important to determine whether the integral converges or diverges.

**Theorem 8.8 (Comparison Test for Improper Integrals)**

*Suppose the function $f$ and $g$ are continuous and $0 \le g(x) \le f(x)$ on the interval $[a, \infty)$. It can be shown that if $\int_a^\infty f(x)\,\mathrm{d}x$ converges, then $\int_a^\infty g(x)\,\mathrm{d}x$ also converges, and if $\int_a^\infty g(x)\,\mathrm{d}x$ diverges, then $\int_a^\infty f(x)\,\mathrm{d}x$ also diverges.*

## Example 12 (Comparison Test for Improper Integrals)

Determine whether $\int_1^\infty e^{-2} \, \mathrm{d}x$ converges of diverges.

$$\int_1^\infty e^{-x} \, \mathrm{d}x = \lim_{b\to\infty} \int_1^b e^{-x} \, \mathrm{d}x = \lim_{b\to\infty} \left[-e^{-x}\right]_1^b = \frac{1}{e}$$

Because $e^{-x^2} \leq e^{-x}$ on $[1, \infty)$ and $\int_1^\infty e^{-x} \, \mathrm{d}x$ converges, then by the comparison test so does $\int_1^\infty e^{-x^2} \, \mathrm{d}x$. ■