Chapter 7 Applications of Integration

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Area of a region between two curves

- With a few modifications, you can extend the application of definite integrals from the area of a region under a curve to the area of a region between two curves.
- Consider two functions f and g that are continuous on the interval $[a, b]$.

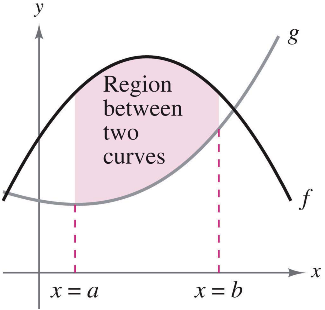

Figure 1: Area of a region between two curves.

- If, as in Figure 1, the graphs of both f and g lie above the x-axis, and the graph of g lies below the graph of f, you can geometrically interpret the area of the region between the graphs as the area of the region under the graph of g subtracted from the area of the region under the graph of f, as shown in Figure 2.

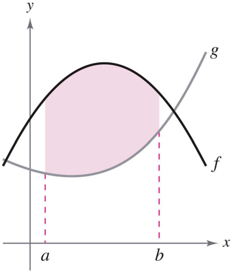

Figure 2: $\int_a^b [f(x) - g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x$

- To verify the reasonableness of the result shown in Figure 2, you can partition the interval $[a, b]$ into n subintervals, each of width Δx.
- Then, as shown in Figure 3, sketch a representative rectangle of width Δx and height $f(x_i) - g(x_i)$, where x_i is in the ith subinterval.

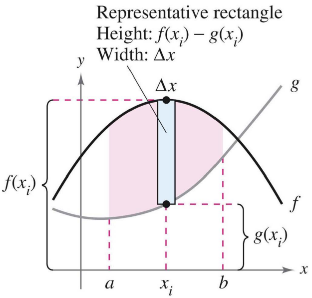

Figure 3: Representative rectangle. Height: $f(x_i) - g(x_i)$; Width: Δx.

- The area of this representative rectangle is

$$\Delta A_i = (\text{height})(\text{width}) = [f(x_i) - g(x_i)]\Delta x.$$

- By adding the areas of the n rectangles and taking the limit as $\|\Delta\| \to 0$ $(n \to \infty)$, you obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x.$$

- Because f and g are continuous on $[a, b]$, $f - g$ is also continuous on $[a, b]$ and the limit exists. So, the area of the given region is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x = \int_a^b [f(x) - g(x)]\,\mathrm{d}x.$$

Area of a region between two curves

If f and g are continuous on $[a,b]$ and $g(x) \leq f(x)$ for all x in $[a,b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,\mathrm{d}x.$$

- In Figure 1, the graphs of f and g are shown above the x-axis. This, however, is not necessary.
- The same integrand $[f(x) - g(x)]$ can be used as long as f and g are continuous and $g(x) \leq f(x)$ for all x in the interval $[a,b]$.
- This is summarized graphically in Figure 4.
- Notice in Figure 4 that the height of a representative rectangle is $f(x) - g(x)$ regardless of the relative position of the x-axis.

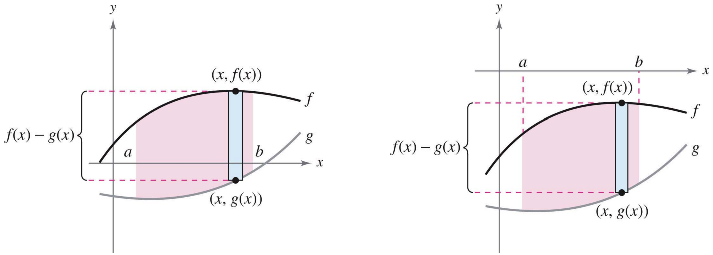

Figure 4: The height of a representative rectangle.

- Representative rectangles are used throughout this chapter in various applications of integration.
- A vertical rectangle (of width Δx) implies integration with respect to x, whereas a horizontal rectangle (of width Δy) implies integration with respect to y.

Example 1 (Finding the area of a region between two curves)

Find the area of the region bounded by the graphs of $f(x) = x^2 + 2$, $g(x) = -x$, $x = 0$, and $x = 1$.

- Let $g(x) = -x$ and $f(x) = x^2 + 2$.
- Then $g(x) \leq f(x)$ for all x in $[0, 1]$, as shown in Figure 5.
- So, the area of the representative rectangle is

$$\Delta A = [f(x) - g(x)]\Delta x = [(x^2 + 2) - (-x)]\Delta x$$

and the area of the region is

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)] \, \mathrm{d}x = \int_0^1 [(x^2 + 2) - (-x)] \, \mathrm{d}x \\ &= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_0^1 = \frac{1}{3} + \frac{1}{2} + 2 = \frac{17}{6}. \end{aligned}$$

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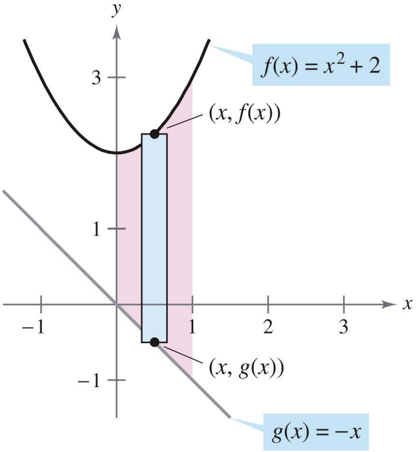

Figure 5: Region bounded by the graph of $f(x) = x^2 + 2$, $g(x) = -x$, $x = 0$, and $x = 1$.

- In Example 1, the graphs of $f(x) = x^2 + 2$ and $g(x) = -x$ do not intersect, and the values of a and b are given explicitly.
- A more common problem involves the area of a region bounded by two intersecting graphs, where the values of a and b must be calculated.

Example 2 (A region lying between two intersecting graphs)

Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

- In Figure 6, notice that the graphs of f and g have two points of intersection.
- To find the x-coordinates of these points, set $f(x)$ and $g(x)$ equal to each other and solve for x.

$$
\begin{aligned}
2 - x^2 &= x \\
-x^2 - x + 2 &= 0 \\
-(x+2)(x-1) &= 0 \\
x &= -2 \quad \text{or} \quad 1
\end{aligned}
$$

So, $a = -2$ and $b = 1$.Because $g(x) \leq f(x)$ for all x in the interval $[-2, 1]$, the representative rectangle has an area of

$$\Delta A = [f(x) - g(x)]\Delta x = [(2 - x^2) - x]\Delta x$$

and the area of the region is

$$A = \int_{-2}^{1} [(2 - x^2) - x] \,\mathrm{d}x = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1} = \frac{9}{2}. \qquad \blacksquare$$

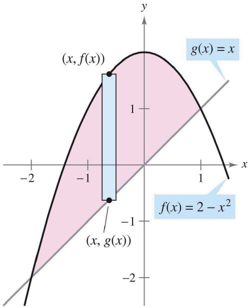

Figure 6: Region bounded by the graph of $f(x) = 2 - x^2$ and the graph of $g(x) = x$.

Example 3 (A region lying between two intersecting graphs)

The sine and cosine curves intersect infinitely many times, bounding regions of equal areas, as shown in Figure 7. Find the area of one of these regions.

$$\sin x = \cos x \qquad \frac{\sin x}{\cos x} = 1 \qquad \tan x = 1 \qquad x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}, \qquad 0 \le x \le 2\pi$$

Because $\sin x \ge \cos x$ for all x in the interval $[\pi/4, 5\pi/4]$, the area of the region is

$$A = \int_{\pi/4}^{5\pi/4} [\sin x - \cos x]\,\mathrm{d}x = [-\cos x - \sin x]_{\pi/4}^{5\pi/4} = 2\sqrt{2}. \qquad \blacksquare$$

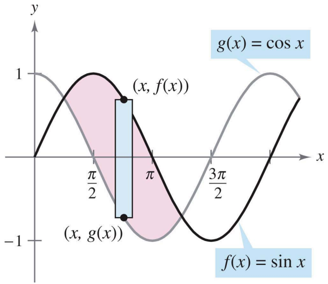

Figure 7: One of the regions bounded by the graphs of the sine and cosine functions.

Example 4 (Curves that intersect at more than two points)

Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.

- Begin by setting $f(x)$ and $g(x)$ equal to each other and solving for x. This yields the x-values at all points of intersection of the two graphs.

$$\begin{aligned} 3x^3 - x^2 - 10x &= -x^2 + 2x \\ 3x^3 - 12x &= 0 \\ 3x(x-2)(x+2) &= 0 \quad x = -2,\ 0,\ 2 \end{aligned}$$

  So, the two graphs intersect when $x = -2$, 0, and 2.

- In Figure 8, notice that $g(x) \le f(x)$ on the interval $[-2, 0]$.
- However, the two graphs switch at the origin, and $f(x) \le g(x)$ on the interval $[0, 2]$.

- So, you need two integrals—one for the interval $[-2, 0]$ and one for the interval $[0, 2]$.

$$
\begin{aligned}
A &= \int_{-2}^{0} [f(x) - g(x)]\,\mathrm{d}x + \int_{0}^{2} [g(x) - f(x)]\,\mathrm{d}x \\
&= \int_{-2}^{0} \left(3x^3 - 12x\right)\,\mathrm{d}x + \int_{0}^{2} \left(-3x^3 + 12x\right)\,\mathrm{d}x \\
&= \left[\frac{3x^4}{4} - 6x^2\right]_{-2}^{0} + \left[-\frac{3x^4}{4} + 6x^2\right]_{0}^{2} = -(12 - 24) + (-12 + 24) \\
&= 24
\end{aligned}
\qquad \blacksquare
$$

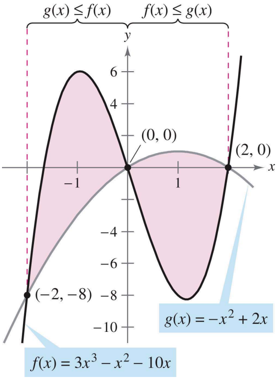

Figure 8: On $[-2, 0]$, $g(x) \leq f(x)$, and on $[0, 2]$, $f(x) \leq g(x)$.

Example 5 (Horizontal representative rectangles)

Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.

Consider $g(y) = 3 - y^2$ and $f(y) = y + 1$. These two curves intersect when $y = -2$ and $y = 1$, as shown in Figure 9. Because $f(y) \le g(y)$ on this interval, you have

$$\Delta A = [g(y) - f(y)]\Delta y = [(3 - y^2) - (y + 1)]\Delta y$$

So, the area is

$$\begin{aligned} A &= \int_{-2}^{1} [(3 - y^2) - (y + 1)]\, \mathrm{d}y = \int_{-2}^{1} (-y^2 - y + 2)\, \mathrm{d}y \\ &= \left[\frac{-y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^{1} \\ &= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}. \end{aligned}$$

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In Example 5, notice that by integrating with respect to y, you need only one integral. To integrate with respect to x, you would need two integrals because the upper boundary changes at $x = 2$, as shown in Figure 9.

$$
\begin{aligned}
A &= \int_{-1}^{2} \left[(x-1) + \sqrt{3-x}\right] \mathrm{d}x + \int_{2}^{3} (\sqrt{3-x} + \sqrt{3-x}) \,\mathrm{d}x \\
&= \int_{-1}^{2} [x - 1 + (3-x)^{1/2}] \,\mathrm{d}x + 2\int_{2}^{3} (3-x)^{1/2} \,\mathrm{d}x \\
&= \left[\frac{x^2}{2} - x - \frac{(3-x)^{3/2}}{3/2}\right]_{-1}^{2} - 2\left[\frac{(3-x)^{3/2}}{3/2}\right]_{2}^{3} \\
&= \left(2 - 2 - \frac{2}{3}\right) - \left(\frac{1}{2} + 1 - \frac{16}{3}\right) - 2(0) + 2\left(\frac{2}{3}\right) = \frac{9}{2}.
\end{aligned}
$$

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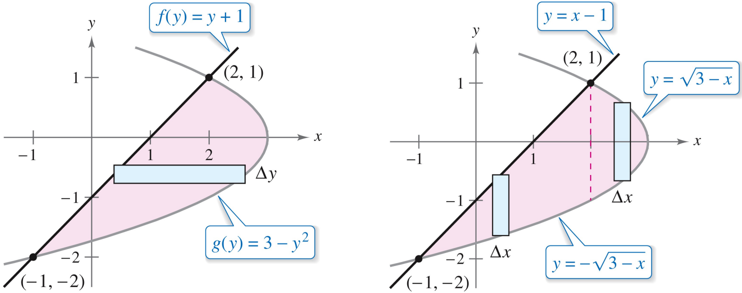

(a) Horizontal rectangles (integration with respect to y)

(b) Vertical rectangles (integration with respect to x)

Figure 9: Horizontal rectangles v.s. vertical rectangles.

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The Disk Method

- If a region in the plane is revolved about a line, the resulting solid is a solid of revolution, and the line is called the axis of revolution.
- The simplest such solid is a right circular cylinder or disk, which is formed by revolving a rectangle about an axis adjacent to one side of the rectangle, as shown in Figure 10.

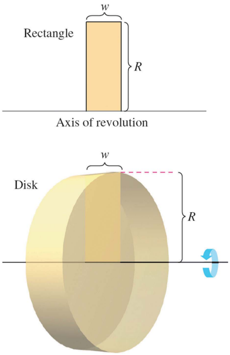


Figure 10: Volume of a disk: $\pi R^2 w$.

- The volume of such a disk is

$$\text{Volume of disk} = (\text{area of disk})(\text{width of disk}) = \pi R^2 w$$

  where R is the radius of the disk and w is the width.
- To see how to use the volume of a disk to find the volume of a general solid of revolution, consider a solid of revolution formed by revolving the plane region in Figure 11 about the indicated axis.

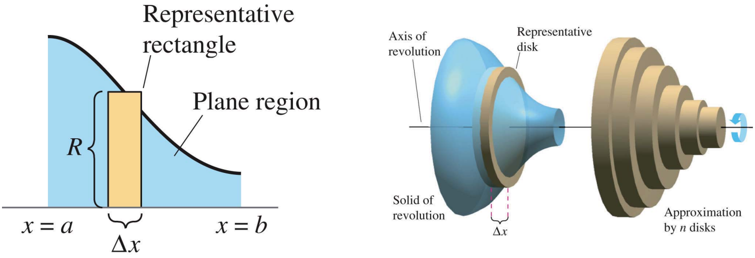


Figure 11: Disk Method.

- To determine the volume of this solid, consider a representative rectangle in the plane region. When this rectangle is revolved about the axis of revolution, it generates a representative disk whose volume is

$$\Delta V = \pi R^2 \Delta x.$$

- Approximating the volume of the solid by n such disks of width Δx and radius $R(x_i)$ produces

$$\text{Volume of solid} \approx \sum_{i=1}^{n} \pi [R(x_i)]^2 \Delta x = \pi \sum_{i=1}^{n} [R(x_i)]^2 \Delta x.$$

- This approximation appears to become better and better as $\|\Delta\| \to 0$ $(n \to \infty)$. So, you can define the volume of the solid as

$$\text{Volume of solid} = \lim_{\|\Delta\| \to 0} \pi \sum_{i=1}^{n} [R(x_i)]^2 \Delta x = \pi \int_a^b [R(x)]^2 \, \mathrm{d}x.$$

- Schematically, the Disk Method looks like this.

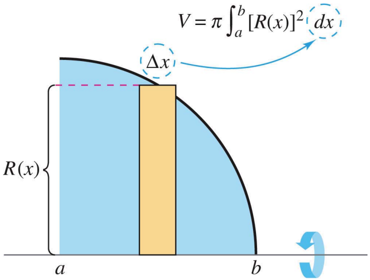
Known precalculus formula		Representative element		New integration formula
Volume of disk $V = \pi R^2 w$	$\Rightarrow$	$\Delta V = \pi [R(x_i)]^2 \Delta x$	$\Rightarrow$	Solid of revolution $V = \pi \int_a^b [R(x)]^2 \, \mathrm{d}x$

- A similar formula can be derived when the axis of revolution is vertical.

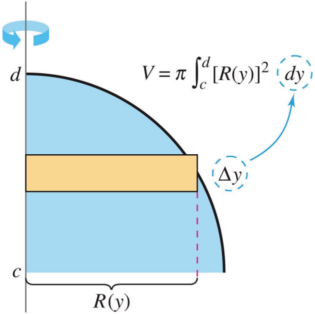
The Disk Method

To find the volume of a solid of revolution with the Disk Method, use one of the following, as shown in Figure 12.

Horizontal axis of revolution	Vertical axis of revolution
Volume $= V = \pi \int_a^b [R(x)]^2 \, \mathrm{d}x$	Volume $= V = \pi \int_c^d [R(y)]^2 \, \mathrm{d}y$



(a) Horizontal axis of revolution.



(b) Vertical axis of revolution.

Figure 12: Find the volume of a solid of revolution with the Disk Method.

Example 1 (Using the Disk Method)

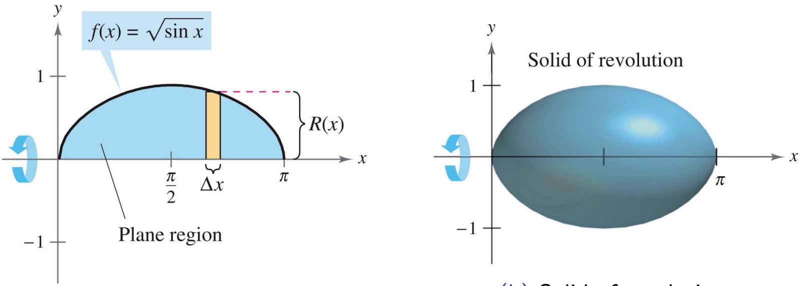
Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = \sqrt{\sin x}$ and the x-axis $(0 \le x \le \pi)$ about the x-axis.

- From the representative rectangle in Figure 13, you can see that the radius of this solid is

$$R(x) = f(x) = \sqrt{\sin x}.$$

- So, the volume of the solid of revolution is

$$\begin{aligned} V &= \pi \int_a^b [R(x)]^2 \, \mathrm{d}x = \pi \int_0^\pi \left(\sqrt{\sin x}\right)^2 \, \mathrm{d}x \\ &= \pi \int_0^\pi \sin x \, \mathrm{d}x = \pi \left[-\cos x\right]_0^\pi = \pi(1+1) = 2\pi. \end{aligned} \qquad \blacksquare$$



(a) Plane region.

(b) Solid of revolution.

Figure 13: Disk Method: $f(x) = \sqrt{\sin x}$.

Example 2 (Revolving about a line that is not a coordinate axis)

Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = 2 - x^2$ and $g(x) = 1$ about the line $y = 1$, as shown in Figure 14.

The two graphs intersect when $x = \pm 1$.

$$R(x) = f(x) - g(x) = (2 - x^2) - 1 = 1 - x^2$$

$$V = \pi \int_a^b [R(x)]^2 \, \mathrm{d}x = \pi \int_{-1}^{1} (1 - x^2)^2 \, \mathrm{d}x$$

$$= \pi \int_{-1}^{1} (1 - 2x^2 + x^4) \, \mathrm{d}x = \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^{1} = \frac{16\pi}{15} \qquad \blacksquare$$

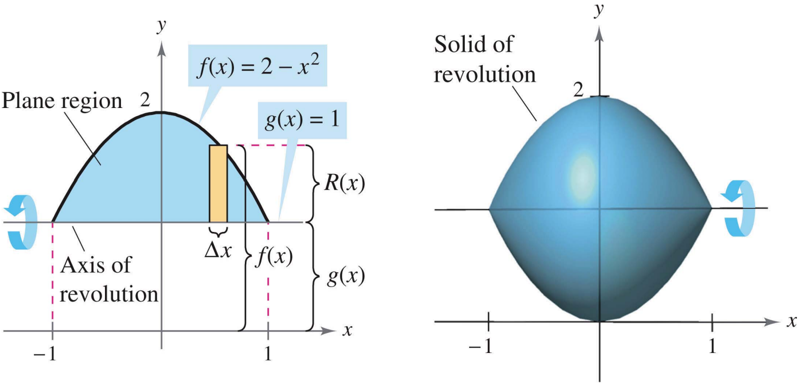


Figure 14: Revolving about a line that is not a coordinate axis.

The Washer Method

- The Disk Method can be extended to cover solids of revolution with holes by replacing the representative disk with a representative washer.
- The washer is formed by revolving a rectangle about an axis, as shown in Figure 15.
- If r and R are the inner and outer radii of the washer and w is the width of the washer, the volume is given by Volume of washer $= \pi(R^2 - r^2)w$.
- To see how this concept can be used to find the volume of a solid of revolution, consider a region bounded by an outer radius $R(x)$ and an inner radius $r(x)$, as shown in Figure 16.

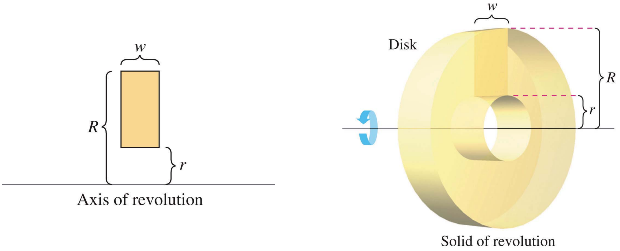


Figure 15: Washer Method.

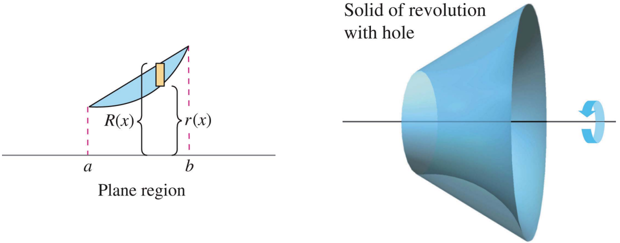


Figure 16: Solid of revolution with hole.

- If the region is revolved about its axis of revolution, the volume of the resulting solid is given by

$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2\right) \mathrm{d}x. \qquad \text{Washer Method}$$

- Note that the integral involving the inner radius represents the volume of the hole and is subtracted from the integral involving the outer radius.

Example 3 (Using the Washer Method)

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x-axis, as shown in Figure 17.

- In Figure 17, you can see that the outer and inner radii are as follows.

$$R(x) = \sqrt{x} \quad \text{Outer radius} \qquad\qquad r(x) = x^2 \quad \text{Inner radius}$$

- Integrating between 0 and 1 produces

$$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2)\,\mathrm{d}x = \pi \int_0^1 [(\sqrt{x})^2 - (x^2)^2]\,\mathrm{d}x$$

$$= \pi \int_0^1 (x - x^4)\,\mathrm{d}x = \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \frac{3\pi}{10}. \qquad \blacksquare$$

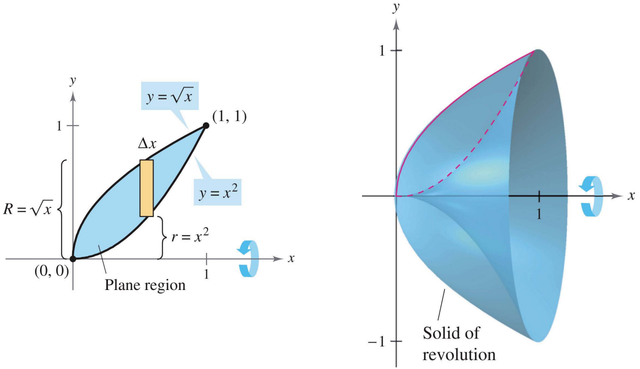


Figure 17: Solid of revolution.

- So far, the axis of revolution has been horizontal and you have integrated with respect to x. In the Example 4, the axis of revolution is vertical and you integrate with respect to y. In this example, you need two separate integrals to compute the volume.

Example 4 (Integrating with respect to y, two-integral case)

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about y-axis, as shown in Figure 18.

- For the region shown in Figure 18, the outer radius is simply $R = 1$.
- There is, however, no convenient formula that represents the inner radius.
- When $0 \leq y \leq 1$, $r = 0$, but when $1 \leq y \leq 2$, r is determined by the equation $y = x^2 + 1$, which implies that $r = \sqrt{y-1}$.

$$r(y) = \begin{cases} 0, & 0 \leq y \leq 1 \\ \sqrt{y-1}, & 1 \leq y \leq 2 \end{cases}$$

- Using this definition of the inner radius, you can use two integrals to find the volume.

$$
\begin{aligned}
V &= \pi \int_0^1 (1^2 - 0^2)\,\mathrm{d}y + \pi \int_1^2 \left[1^2 - \left(\sqrt{y-1}\right)^2\right] \mathrm{d}y \\
&= \pi \int_0^1 1\,\mathrm{d}y + \pi \int_1^2 (2-y)\,\mathrm{d}y = \pi\,[y]_0^1 + \pi \left[2y - \frac{y^2}{2}\right]_1^2 \\
&= \pi + \pi\left(4 - 2 - 2 + \frac{1}{2}\right) = \frac{3\pi}{2}
\end{aligned}
$$

- Note that the first integral $\pi \int_0^1 1\,\mathrm{d}y$ represents the volume of a right circular cylinder of radius 1 and height 1.
- This portion of the volume could have been determined without using calculus. ■

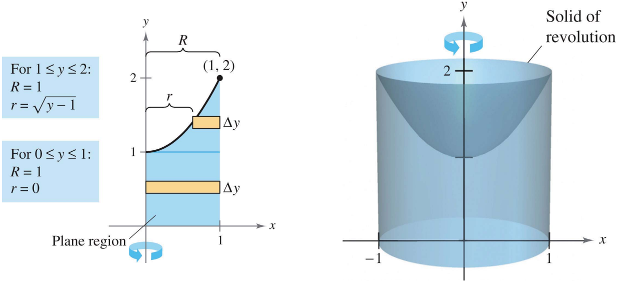


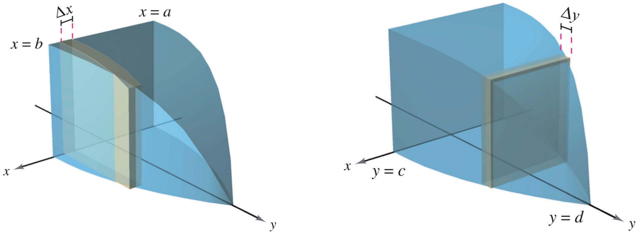
Figure 18: The volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about y-axis.

Solids with known cross sections

- With the Disk Method, you can find the volume of a solid having a circular cross section whose area is $A = \pi R^2$.
- This method can be generalized to solids of any shape, as long as you know a formula for the area of an arbitrary cross section.
- Some common cross sections are squares, rectangles, triangles, semicircles, and trapezoids.

Volumes of solids with known cross sections

1. For cross sections of area $A(x)$ taken perpendicular to the x-axis,
$$\text{Volume} = \int_a^b A(x)\,\mathrm{d}x. \qquad \text{See Figure 19}$$
2. For cross sections of area $A(y)$ taken perpendicular to the y-axis,
$$\text{Volume} = \int_c^d A(y)\,\mathrm{d}y. \qquad \text{See Figure 19}$$



(a) Cross sections perpendicular to x-axis.

(b) Cross sections perpendicular to y-axis.

Figure 19: Solid with known cross sections.

Example 6 (Triangular cross sections)

Find the volume of the solid shown in Figure 20. The base of the solid is the region bounded by the lines $f(x) = 1 - \frac{x}{2}$, $g(x) = -1 + \frac{x}{2}$, and $x = 0$. The cross sections perpendicular to the x-axis are equilateral triangles.

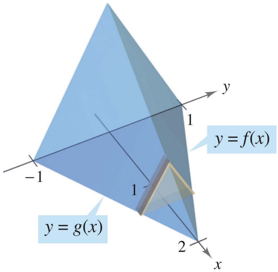
- The base and area of each triangular cross section are as follows.

$$\begin{aligned}\text{Base} &= \left(1-\frac{x}{2}\right)-\left(-1+\frac{x}{2}\right)=2-x\\ \text{Area} &= \frac{\sqrt{3}}{4}(\text{base})^2\\ A(x) &= \frac{\sqrt{3}}{4}(2-x)^2\end{aligned}$$

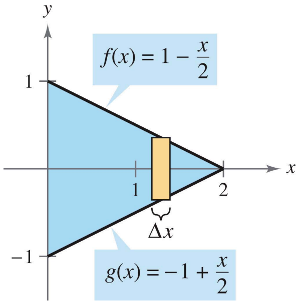
- Because x ranges from 0 to 2, the volume of the solid is

$$V=\int_a^b A(x)\,\mathrm{d}x=\int_0^2 \frac{\sqrt{3}}{4}(2-x)^2\,\mathrm{d}x=-\frac{\sqrt{3}}{4}\left[\frac{(2-x)^3}{3}\right]_0^2=\frac{2\sqrt{3}}{3}.$$

■



(a) Cross sections are equilateral triangles.



(b) Triangular base in xy-plane.

Figure 20: Equilateral triangle cross section.

Table of Contents

Arc length

- Definite integrals are use to find the arc length of curves and the areas of surfaces of revolution.
- In either case, an arc (a segment of a curve) is approximated by straight line segments whose lengths are given by the familiar Distance Formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

- A rectifiable curve is one that has a finite arc length.
- You will see that a sufficient condition for the graph of a function f to be rectifiable between $(a, f(a))$ and $(b, f(b))$ is that f' be continuous on $[a, b]$.
- Such a function is continuously differentiable on $[a, b]$, and its graph on the interval $[a, b]$ is a smooth curve.

- Consider a function $y = f(x)$ that is continuously differentiable on the interval $[a, b]$. You can approximate the graph of f by n line segments whose endpoints are determined by the partition $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ as shown in Figure 21.

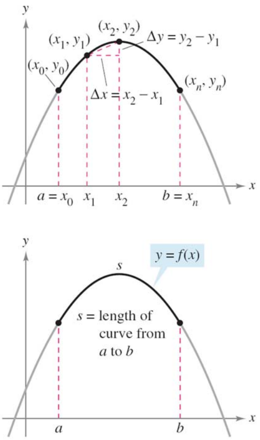


Figure 21: Arc length.

Definition 7.1 (Arc length)

Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The arc length of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\,\mathrm{d}x.$$

Similarly, for a smooth curve given by $x = g(y)$, the arc length of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\,\mathrm{d}y.$$

Example 1 (The length of a line segment)

Find the arc length from (x_1, y_1) to (x_2, y_2) on the graph of $f(x) = mx + b$, as shown in Figure 22.

Because $m = f'(x) = \frac{y_2 - y_1}{x_2 - x_1}$ it follows that

$$s = \int_{x_1}^{x_2} \sqrt{1 + [f'(x)]^2}\,\mathrm{d}x = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2}\,\mathrm{d}x$$

$$= \sqrt{\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{(x_2 - x_1)^2}}\,(x)\,\Big]_{x_1}^{x_2}$$

$$= \sqrt{\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{(x_2 - x_1)^2}}(x_2 - x_1) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

which is the formula for the distance between two points in the plane. ■

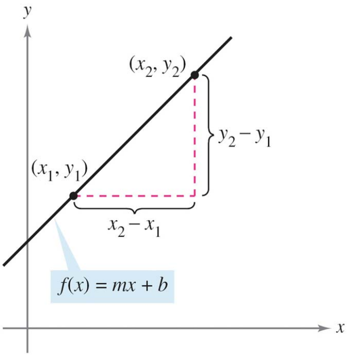


Figure 22: The formula for the arc length of the graph of $f(x) = mx + b$ from (x_1, y_1) to (x_2, y_2) is the same as the standard Distance Formula.

Example 2 (Finding arc length)

Find the arc length of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $\left[\frac{1}{2}, 2\right]$, as shown in Figure 23.

Using

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$$

yields an arc length of

$$\begin{aligned} s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x &= \int_{1/2}^{2} \sqrt{1 + \left[\frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)\right]^2}\,\mathrm{d}x \\ &= \int_{1/2}^{2} \sqrt{\frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right)}\,\mathrm{d}x \\ &= \int_{1/2}^{2} \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)\,\mathrm{d}x = \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_{1/2}^{2} \\ &= \frac{33}{16}. \end{aligned}$$

■

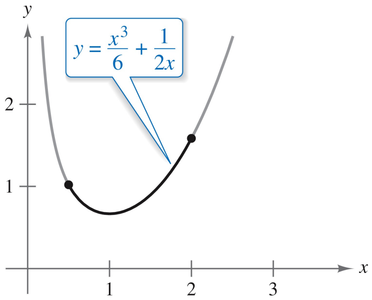


Figure 23: The arc length of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$ on $\left[\frac{1}{2}, 2\right]$.

Example 4 (Finding arc length)

Find the arc length of the graph of $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$, as shown in Figure 24.

Using $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sin x}{\cos x} = -\tan x$ yields an arc length of

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\,\mathrm{d}x = \int_0^{\pi/4} \sqrt{\sec^2 x}\,\mathrm{d}x$$

$$= \int_0^{\pi/4} \sec x\,\mathrm{d}x = [\ln|\sec x + \tan x|]_0^{\pi/4} = \ln(\sqrt{2} + 1) - \ln 1 \approx 0.881.$$

■

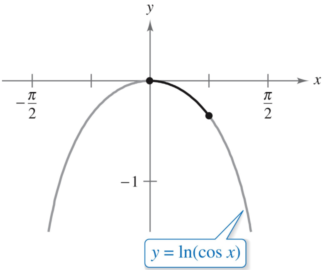


Figure 24: The arc length of the graph of y on $\left[0, \frac{\pi}{4}\right]$.

Definition 7.2 (Surface of revolution)

If the graph of a continuous function is revolved about a line, the resulting surface is a surface of revolution.

- The area of a surface of revolution is derived from the formula for the lateral surface area of the frustum of a right circular cone.
- Consider the line segment in Figure 25, where L is the length of the line segment, r_1 is the radius at the left end of the line segment, and r_2 is the radius at the right end of the line segment.

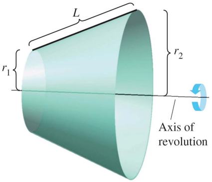


Figure 25: Surface of revolution.

- When the line segment is revolved about its axis of revolution, it forms a frustum of a right circular cone, with

$$S = 2\pi r L \qquad \text{Lateral surface area of frustum}$$

where

$$r = \frac{1}{2}(r_1 + r_2). \qquad \text{Average radius of frustum}$$

- Suppose the graph of a function f, having a continuous derivative on the interval $[a, b]$, is revolved about the x-axis to form a surface of revolution, as shown in Figure 26.

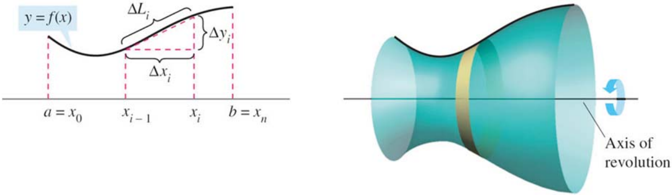


Figure 26: Surface of revolution.

- Let $\triangle$ be a partition of $[a, b]$, with subintervals of width $\triangle x_i$. Then the line segment of length $\triangle L_i = \sqrt{\triangle x_i^2 + \triangle y_i^2}$ generates a frustum of a cone.
- Let r_i be the average radius of this frustum. By the Intermediate Value Theorem, a point d_i exists (in the ith subinterval) such that $r_i = f(d_i)$. The lateral surface area $\triangle S_i$ of the frustum is

$$\triangle S_i = 2\pi r_i \triangle L_i = 2\pi f(d_i)\sqrt{\triangle x_i^2 + \triangle y_i^2} = 2\pi f(d_i)\sqrt{1 + \left(\frac{\triangle y_i}{\triangle x_i}\right)^2}\triangle x_i.$$

- By the Mean Value Theorem, a number c_i exists in (x_{i-1}, x_i) such that

$$f'(c_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{\triangle y_i}{\triangle x_i}.$$

- So, $\triangle S_i = 2\pi f(d_i)\sqrt{1 + [f'(c_i)]^2}\triangle x_i$, and the total surface area can be approximated by

$$S \approx 2\pi \sum_{i=1}^{n} f(d_i)\sqrt{1 + [f'(c_i)]^2}\triangle x_i.$$

- It can be shown that the limit of the right side as $\|\triangle\| \to 0$ $(n \to \infty)$ is

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,\mathrm{d}x.$$

- In a similar manner, if the graph of f is revolved about the y-axis, then S is

$$S = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}\,\mathrm{d}x.$$

- In these two formulas for S, you can regard the products $2\pi f(x)$ and $2\pi x$ as the circumferences of the circles traced by a point (x, y) on the graph of f as it is revolved about the x-axis and the y-axis (Figure 27). In one case the radius is $r = f(x)$, and in the other case the radius is $r = x$.

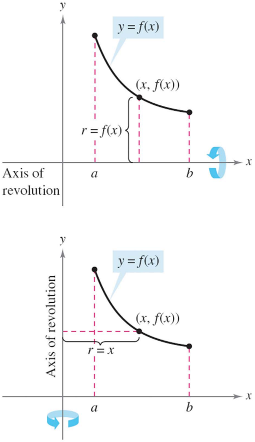


Figure 27: Revolve about x-axis and y-axis.

Definition 7.3 (Area of a surface of revolution)

Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\,\mathrm{d}x \qquad y \text{ is a function of } x$$

where $r(x)$ is the distance between the graph of f and the axis of revolution. If $x = g(y)$ on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_c^d r(y)\sqrt{1 + [g'(y)]^2}\,\mathrm{d}y \qquad x \text{ is a function of } y$$

where $r(y)$ is the distance between the graph of g and the axis of revolution.

Example 5 (The area of a surface of revolution)

Find the area of the surface formed by revolving the graph of $f(x) = x^3$ on the interval $[0, 1]$ about the x-axis, as shown in Figure 28.

The distance between the x-axis and the graph of f is $r(x) = f(x)$, and because $f'(x) = 3x^2$, the surface area is

$$\begin{aligned} S &= 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\,\mathrm{d}x = 2\pi \int_0^1 x^3\sqrt{1 + (3x^2)^2}\,\mathrm{d}x \\ &= \frac{2\pi}{36}\int_0^1 (36x^3)(1 + 9x^4)^{1/2}\,\mathrm{d}x \\ &= \frac{\pi}{18}\left[\frac{(1 + 9x^4)^{3/2}}{3/2}\right]_0^1 = \frac{\pi}{27}(10^{3/2} - 1) \approx 3.563. \end{aligned}$$

■

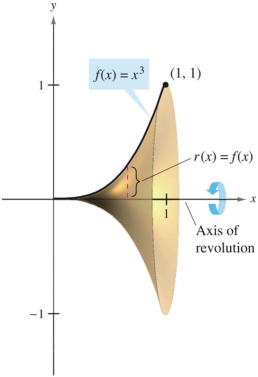


Figure 28: Area of a surface of revolution: $f(x) = x^3$ about x-axis.

Example 6 (The area of a surface of revolution)

Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval $[0, \sqrt{2}]$ about the y-axis, as shown in Figure 29.

- In this case, the distance between the graph of f and the y-axis is $r(x) = x$.
- Using $f'(x) = 2x$, you can determine that the surface area is

$$\begin{aligned} S &= 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\,\mathrm{d}x = 2\pi \int_0^{\sqrt{2}} x\sqrt{1 + (2x)^2}\,\mathrm{d}x \\ &= \frac{2\pi}{8} \int_0^{\sqrt{2}} (1 + 4x^2)^{1/2}(8x)\,\mathrm{d}x = \frac{\pi}{4}\left[\frac{(1 + 4x^2)^{3/2}}{3/2}\right]_0^{\sqrt{2}} \\ &= \frac{\pi}{6}\left[(1 + 8)^{3/2} - 1\right] = \frac{13\pi}{3} \approx 13.614. \end{aligned}$$

■

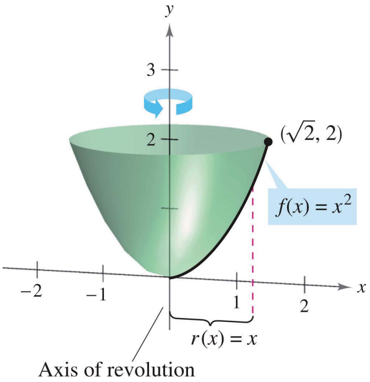


Figure 29: The area of a surface revolution: $f(x) = x^2$ about y-axis.