Chapter 5 Logarithmic, Exponential, and Other Transcendental Functions

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The natural logarithmic function

- The General Power Rule

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \qquad \text{General Power Rule}$$

  has an important disclaimer—it doesn't apply when $n = -1$. Consequently, you have not yet found an antiderivative for the function $f(x) = 1/x$.
- In this section, you will use the Second Fundamental Theorem of Calculus to define such a function.
- This antiderivative is a function that you have not encountered previously in the text.
- It is neither algebraic nor trigonometric, but falls into a new class of functions called logarithmic functions.
- This particular function is the natural logarithmic function.

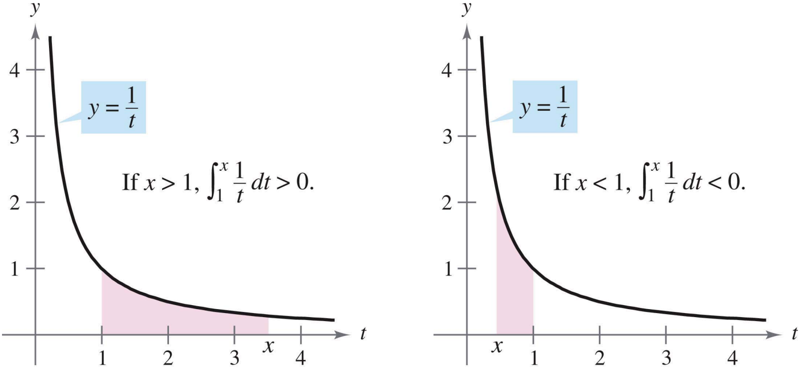
Definition 5.1 (The natural logarithmic function)

The natural logarithmic function is defined by

$$\ln x = \int_1^x \frac{1}{t}\,\mathrm{d}t, \quad x > 0.$$

The domain of the natural logarithmic function is the set of all positive real numbers.

- From this definition, you can see that $\ln x$ is positive for $x > 1$ and negative for $0 < x < 1$, as shown in Figure 1.
- Moreover, $\ln(1) = 0$, because the upper and lower limits of integration are equal when $x = 1$.



(a) If $x > 1$, then $\ln x > 0$. (b) If $0 < x < 1$, then $\ln x < 0$.

Figure 1: The natural logarithmic function $\ln x$.

- To sketch the graph of $y = \ln x$, you can think of the natural logarithmic function as an antiderivative given by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}.$$

- Figure 2 is a computer-generated graph, called a slope (or direction) field, showing small line segments of slope $1/x$.
- The graph of $y = \ln x$ is the solution that passes through the point $(1, 0)$.

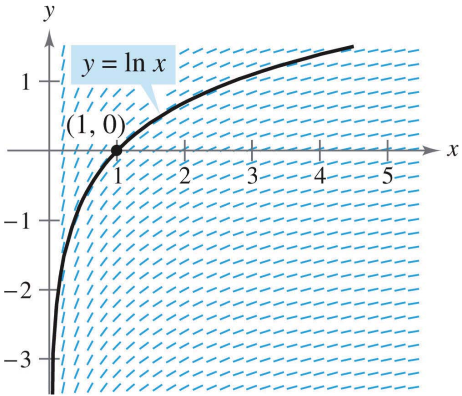


Figure 2: Each small line segment has a slope of $\frac{1}{x}$.

Theorem 5.1 (Properties of the natural logarithmic function)

The natural logarithmic function has the following properties.

1. *The domain is* $(0, \infty)$ *and the range is* $(-\infty, \infty)$.
2. *The function is continuous, increasing, and one-to-one.*
3. *The graph is concave downward.*

- The domain is $f(x) = \ln x$ is $(0, \infty)$ by definition.
- Moreover, the function is continuous because it is differentiable.
- It is increasing because derivative

$$f'(x) = \frac{1}{x}$$

  is positive for $x > 0$, as shown in Figure 3.
- It is concave downward because

$$f''(x) = -\frac{1}{x^2}$$

  is negative for $x > 0$.

- Recall from Section P.3 that a function f is one-to-one if for x_1 and x_2 in its domain

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2).$$

- Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x} > 0$ for $x > 0$.
- So f is increasing on its entire domain $(0, \infty)$ and therefore is strictly monotonic.
- Choose x_1 and x_2 in the domain of f such that $x_1 \neq x_2$. Because f is strictly monotonic, it follows that either

$$f(x_1) < f(x_2) \quad \text{or} \quad f(x_1) > f(x_2).$$

- In either case, $f(x_1) \neq f(x_2)$. So, $f(x) = \ln x$ is one-to-one. To verify the limits, begin by showing that $\ln 2 \geq \frac{1}{2}$.
- From the Mean Value Theorem for Integrals, you can write

$$\ln 2 = \int_1^2 \frac{1}{x}\,\mathrm{d}x = \frac{1}{c}(2-1) = \frac{1}{c}$$

where c is in $[1, 2]$.

- This implies that

$$1 \le c \le 2 \qquad 1 \ge \frac{1}{c} \ge \frac{1}{2} \qquad 1 \ge \ln 2 \ge \frac{1}{2}.$$

- Now, let N be any positive (large) number. Because $\ln x$ is increasing, it follows that if $x > 2^{2N}$, then

$$\ln x > \ln 2^{2N} = 2N \ln 2.$$

- However, because $\ln 2 \ge \frac{1}{2}$, it follows that

$$\ln x > 2N \ln 2 \ge 2N \left(\frac{1}{2}\right) = N.$$

- This verifies the second limit. To verify the first limit, let $z = 1/x$. Then, $z \to \infty$ as $x \to 0^+$, and you can write

$$\lim_{x \to 0^+} \ln x = \lim_{x \to 0^+} \left(-\ln \frac{1}{x}\right) = \lim_{z \to \infty} (-\ln z) = -\lim_{z \to \infty} \ln z = -\infty.$$

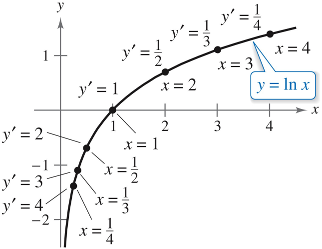


Figure 3: The natural logarithmic function is increasing, and its graph is concave downward.

Theorem 5.2 (Logarithmic properties)

If a and b are positive numbers and n is rational, then the following properties are true.

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

- The first property has already been discussed.
- The proof of the second property follows from the fact that two antiderivatives of the same function differ at most by a constant.
- From the Second Fundamental Theorem of Calculus and the definition of the natural logarithmic function, you know that

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{\mathrm{d}}{\mathrm{d}x} \left[\int_1^x \frac{1}{t} \, \mathrm{d}t \right] = \frac{1}{x}.$$

- So, consider the two derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x} [\ln(ax)] = \frac{a}{ax} = \frac{1}{x}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x} [\ln a + \ln x] = 0 + \frac{1}{x} = \frac{1}{x}.$$

- Because $\ln(ax)$ and $(\ln a + \ln x)$ are both antiderivatives of $1/x$, they must differ at most by a constant.

$$\ln(ax) = \ln x + \ln a + C$$

- By letting $x = 1$, you can see that $C = 0$.

- The third property can be proved similarly by comparing the derivatives of $\ln(x^n)$ and $n \ln x$.
- Finally, using the second and third properties, you can prove the fourth property.

$$\ln\left(\frac{a}{b}\right) = \ln[a(b^{-1})] = \ln a - \ln b$$

$\square$

- When using the properties of logarithms to rewrite logarithmic functions, you must check to see whether the domain of the rewritten function is the same as the domain of the original.
- For instance, the domain of $f(x) = \ln x^2$ is all real numbers except $x = 0$, and the domain of $g(x) = 2 \ln x$ is all positive real numbers. (See Figure 4.)

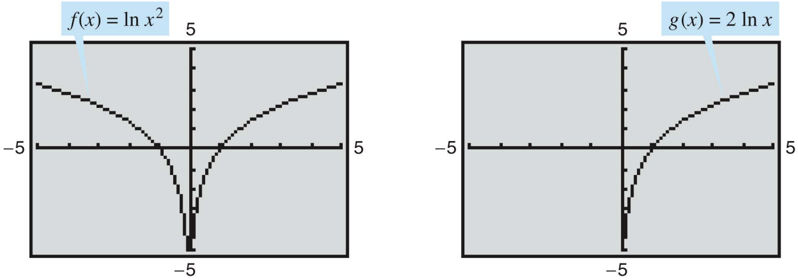


Figure 4: Domain of $f(x) = \ln x^2$ and $g(x) = 2\ln x$.

The number e

- It is likely that you have studied logarithms in an algebra course. There, without the benefit of calculus, logarithms would have been defined in terms of a base number.
- For example, common logarithms have a base of 10 and therefore $\log_{10} 10 = 1$.
- The base for the natural logarithm is defined using the fact that the natural logarithmic function is continuous, is one-to-one, and has a range of $(-\infty, \infty)$.
- So, there must be a unique real number x such that $\ln x = 1$, as shown in Figure 5.
- This number is denoted by the letter e. It can be shown that e is irrational and has the following decimal approximation.

$$e \approx 2.71828182846$$

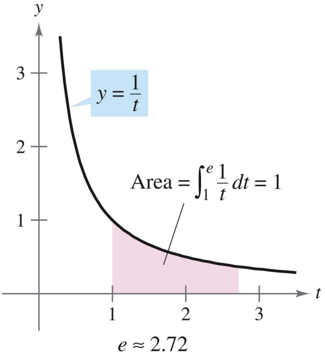


Figure 5: e is the base for the natural logarithm because $\ln e = 1$.

Definition 5.2 (e)

The letter e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t}\,\mathrm{d}t = 1.$$

- Once you know that $\ln e = 1$, you can use logarithmic properties to evaluate the natural logarithms of several other numbers.
- For example, by using the property

$$\ln(e^n) = n \ln e = n(1) = n$$

you can evaluate $\ln(e^n)$ for various values of n as shown in the table and in Figure 6.

x	$\frac{1}{e^3} \approx 0.050$	$\frac{1}{e^2} \approx 0.135$	$\frac{1}{e} \approx 0.368$	$e^0 = 1$	$e \approx 2.718$	$e^2 \approx 7.389$
$\ln x$	-3	-2	-1	0	1	2

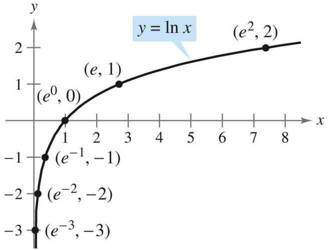


Figure 6: If $x = e^n$, then $\ln x = n$.

- The logarithms shown in the table above are convenient because the x-values are integer powers of e.

Euler's Formula

$$e^{ix} = \cos x + i \sin x$$

Euler's Identity: The most beautiful theorem in mathematics.

$$e^{i\pi} + 1 = 0$$

Example 2 (Evaluating natural logarithmic expressions)

a. $\ln 2 \approx 0.693$ **b.** $\ln 32 \approx 3.466$ **c.** $\ln 0.1 \approx -2.303$ ■

The derivative of the natural logarithmic function

- The derivative of the natural logarithmic function is given in Theorem 5.3.
- The first part of the theorem follows from the definition of the natural logarithmic function as an antiderivative.
- The second part of the theorem is simply the Chain Rule version of the first part.

Theorem 5.3 (Derivative of the natural logarithmic function)

Let u be a differentiable function of x.

1. $\frac{\mathrm{d}}{\mathrm{d}x}[\ln x] = \frac{1}{x}$, $x > 0$ **2.** $\frac{\mathrm{d}}{\mathrm{d}x}[\ln u] = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u'}{u}$, $u > 0$

Example 3 (Differentiation of logarithmic functions)

a. $\frac{\mathrm{d}}{\mathrm{d}x}[\ln(2x)] = \frac{u'}{u} = \frac{2}{2x} = \frac{1}{x},\ u = 2x$

b. $\frac{\mathrm{d}}{\mathrm{d}x}[\ln(x^2+1)] = \frac{u'}{u} = \frac{2x}{x^2+1},\ u = x^2+1$

c. $\frac{\mathrm{d}}{\mathrm{d}x}[x\ln x] = x\left(\frac{\mathrm{d}}{\mathrm{d}x}[\ln x]\right) + (\ln x)\left(\frac{\mathrm{d}}{\mathrm{d}x}[x]\right) = x\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x$

d. $\frac{\mathrm{d}}{\mathrm{d}x}[(\ln x)^3] = 3(\ln x)^2 \frac{\mathrm{d}}{\mathrm{d}x}[\ln x] = 3(\ln x)^2 \frac{1}{x}$ ■

Example 4 (Logarithmic properties as aids to differentiation)

Differentiate $f(x) = \ln\sqrt{x+1}$.

$$f(x) = \ln\sqrt{x+1} = \ln(x+1)^{1/2} = \frac{1}{2}\ln(x+1)$$

$$f'(x) = \frac{1}{2}\left(\frac{1}{x+1}\right) = \frac{1}{2(x+1)}$$ ■

Example 5 (Logarithmic properties as aids to differentiation)

Differentiate $f(x) = \ln \frac{x(x^2+1)^2}{\sqrt{2x^3-1}}$.

$$f(x) = \ln \frac{x(x^2+1)^2}{\sqrt{2x^3-1}} = \ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1)$$

$$f'(x) = \frac{1}{x} + 2\left(\frac{2x}{x^2+1}\right) - \frac{1}{2}\left(\frac{6x^2}{2x^3-1}\right) = \frac{1}{x} + \frac{4x}{x^2+1} - \frac{3x^2}{2x^3-1} \quad \blacksquare$$

- It is convenient to use logarithms as aids in differentiating nonlogarithmic functions.
- This procedure is called logarithmic differentiation.

Example 6 (Logarithmic differentiation)

Find the derivative of

$$y = \frac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2.$$

- Note that $y > 0$ for all $x \neq 2$. So, $\ln y$ is defined. Begin by taking the natural logarithm of each side of the equation.
- Then apply logarithmic properties and differentiate implicitly. Finally, solve for y'.

$$y = \frac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2 \qquad \ln y = \ln \frac{(x-2)^2}{\sqrt{x^2+1}}$$

$$\begin{aligned}
\ln y &= 2\ln(x-2) - \frac{1}{2}\ln(x^2+1) \\
\frac{y'}{y} &= 2\left(\frac{1}{x-2}\right) - \frac{1}{2}\left(\frac{2x}{x^2+1}\right) \\
&= \frac{x^2+2x+2}{(x-2)(x^2+1)}
\end{aligned}$$

$$
\begin{aligned}
y' &= y\left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right] \\
&= \frac{(x-2)^2}{\sqrt{x^2+1}}\left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right] \\
&= \frac{(x-2)(x^2+2x+2)}{(x^2+1)^{3/2}}
\end{aligned}
$$

■

Theorem 5.4 (Derivative involving absolute value)

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln|u| = \frac{u'}{u}.$$

- If $u > 0$, then $|u| = u$, and the result follows from Theorem 5.3.
- If $u < 0$, then $|u| = -u$, and you have

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln|u| = \frac{\mathrm{d}}{\mathrm{d}x} \ln(-u) = \frac{-u'}{-u} = \frac{u'}{u}.$$

Example 7 (Derivative involving absolute value)

Find the derivative of

$$f(x) = \ln|\cos x|.$$

Using Theorem 5.4, let $u = \cos x$ and write

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln|\cos x|] = \frac{u'}{u} = \frac{-\sin x}{\cos x} = -\tan x \quad \frac{\mathrm{d}}{\mathrm{d}x}[\ln|u|] = \frac{u'}{u} \quad u = \cos x. \quad \blacksquare$$

Example 8 (Finding relative extrema)

Locate the relative extrema of

$$y = \ln(x^2 + 2x + 3).$$

Differentiating y, you obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x+2}{x^2+2x+3}.$$

Because $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = -1$, you can apply the First Derivative Test and conclude that the point $(-1, \ln 2)$ is a relative minimum. $\blacksquare$

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Log Rule for integration

The differentiation rules

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln|x|\right] = \frac{1}{x} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln|u|\right] = \frac{u'}{u}$$

produce the following integration rule.

Theorem 5.5 (Log Rule for integration)

Let u be a differentiable function of x.
1. $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$ **2.** $\int \frac{1}{u}\,\mathrm{d}u = \ln|u| + C$

Because $\mathrm{d}u = u'\,\mathrm{d}x$, the second formula can also be written as

$$\int \frac{u'}{u}\,\mathrm{d}x = \ln|u| + C. \qquad \text{Alternative form of Log Rule}$$

Example 1 (Using the Log Rule for integration)

$$\int \frac{2}{x}\,\mathrm{d}x = 2\int \frac{1}{x}\,\mathrm{d}x = 2\ln|x| + C = \ln(x^2) + C$$

Because x^2 cannot be negative, the absolute value notation is unnecessary in the final form of the antiderivative. ■

Example 2 (Using the log rule with a change of variables)

Find $\int \frac{1}{4x-1}\,\mathrm{d}x$.

If you let $u = 4x - 1$, then $\mathrm{d}u = 4\,\mathrm{d}x$.

$$\begin{aligned}\int \frac{1}{4x-1}\,\mathrm{d}x &= \frac{1}{4}\int \left(\frac{1}{4x-1}\right) 4\,\mathrm{d}x = \frac{1}{4}\int \frac{1}{u}\,\mathrm{d}u \\ &= \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln|4x-1| + C\end{aligned}$$

■

Example 3 (Finding area with the log rule)

Find the area of the region bounded by the graph of $y = \frac{x}{x^2+1}$ the x-axis, and the lines $x = 0$ and $x = 3$.

If you let $u = x^2 + 1$, then $u' = 2x$.

$$\int_0^3 \frac{x}{x^2+1}\,\mathrm{d}x = \frac{1}{2}\int_0^3 \left(\frac{2x}{x^2+1}\right)\mathrm{d}x$$
$$= \frac{1}{2}\left[\ln(x^2+1)\right]_0^3 = \frac{1}{2}(\ln 10 - \ln 1) = \frac{1}{2}\ln 10 \approx 1.151 \quad \blacksquare$$

Example 4 (Recognizing quotient forms of the Log Rule)

a. $\int \frac{3x^2+1}{x^3+x}\,\mathrm{d}x = \ln|x^3+x| + C, \quad u = x^3 + x$

b. $\int \frac{\sec^2 x}{\tan x}\,\mathrm{d}x = \ln|\tan x| + C, \quad u = \tan x$

c. $\int \frac{x+1}{x^2+2x}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x+2}{x^2+2x}\,\mathrm{d}x = \frac{1}{2}\ln|x^2+2x| + C, \quad u = x^2 + 2x$

d. $\int \frac{1}{3x+2}\,\mathrm{d}x = \frac{1}{3}\int \frac{3}{3x+2}\,\mathrm{d}x = \frac{1}{3}\ln|3x+2| + C, \quad u = 3x + 2 \quad \blacksquare$

- If a rational function has a numerator of degree greater than or equal to that of the denominator, division may reveal a form to which you can apply the Log Rule.
- This is shown in Example 5.

Example 5 (Using long division before integrating)

Find $\int \frac{x^2+x+1}{x^2+1}\,\mathrm{d}x$.

- Begin by using long division to rewrite the integrand.

$$\frac{x^2+x+1}{x^2+1} = 1 + \frac{x}{x^2+1}$$

- Now, you can integrate to obtain

$$\begin{aligned}\int \frac{x^2+x+1}{x^2+1}\,\mathrm{d}x &= \int \left(1 + \frac{x}{x^2+1}\right)\mathrm{d}x \\ &= \int \mathrm{d}x + \frac{1}{2}\int \frac{2x}{x^2+1}\,\mathrm{d}x = x + \frac{1}{2}\ln(x^2+1) + C.\end{aligned}$$

- Check this result by differentiating to obtain the original integrand. ■

Example 6 (Change of variables with the Log Rule)

Find $\int \frac{2x}{(x+1)^2}\,\mathrm{d}x$.

If you let $u = x + 1$, then $\mathrm{d}u = \mathrm{d}x$ and $x = u - 1$.

$$
\begin{aligned}
\int \frac{2x}{(x+1)^2}\,\mathrm{d}x &= \int \frac{2(u-1)}{u^2}\,\mathrm{d}u = 2\int \left(\frac{u}{u^2} - \frac{1}{u^2}\right)\mathrm{d}u \\
&= 2\int \frac{\mathrm{d}u}{u} - 2\int u^{-2}\,\mathrm{d}u \\
&= 2\ln|u| - 2\left(\frac{u^{-1}}{-1}\right) + C \\
&= 2\ln|u| + \frac{2}{u} + C = 2\ln|x+1| + \frac{2}{x+1} + C \qquad \blacksquare
\end{aligned}
$$

The following are guidelines you can use for integration.

Guidelines for integration

1. Learn a basic list of integration formulas. (Including those given in this section, you now have 12 formulas: the Power Rule, the Log Rule, and ten trigonometric rules. By the end of Section 8, this list will have expanded to 20 basic rules.)
2. Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of u that will make the integrand conform to the formula.
3. If you cannot find a u-substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division. Be creative.
4. If you have access to computer software that will find antiderivatives symbolically, use it.
5. Check your result by differentiating to obtain the original integrand.

Example 7 (u-Substitution and the Log Rule)

Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x \ln x}$.

- The solution can be written as an indefinite integral.

$$y = \int \frac{1}{x \ln x} \, \mathrm{d}x$$

- Because the integrand is a quotient whose denominator is raised to the first power, you should try the Log Rule.
- There are three basic choices for u. The choices $u = x$ and $u = x \ln x$ fail to fit the u'/u form of the Log Rule.
- However, the third choice does fit. Letting $u = \ln x$ produces $u' = 1/x$, and you obtain the following.

$$\int \frac{1}{x \ln x} \, \mathrm{d}x = \int \frac{1/x}{\ln x} \, \mathrm{d}x = \int \frac{u'}{u} \, \mathrm{d}x = \ln |u| + C = \ln |\ln x| + C$$

- So, the solution is $y = \ln |\ln x| + C$. ■

Example 8 (Using a trigonometric identity)

Find $\int \tan x \, \mathrm{d}x$.

- This integral does not seem to fit any formulas on our basic list.
- However, by using a trigonometric identity, you obtain

$$\int \tan x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \, \mathrm{d}x.$$

- Knowing that $D_x[\cos x] = -\sin x$, you can let $u = \cos x$ and write

$$\begin{aligned} \int \tan x \, \mathrm{d}x &= -\int \frac{-\sin x}{\cos x} \, \mathrm{d}x = -\int \frac{u'}{u} \, \mathrm{d}x \\ &= -\ln|u| + C = -\ln|\cos x| + C. \end{aligned}$$

■

Example 9 (Derivation of the Secant Formula)

Find $\int \sec x \, \mathrm{d}x$.

- Consider the following procedure.

$$\int \sec x \, \mathrm{d}x = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) \mathrm{d}x = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, \mathrm{d}x$$

- Letting u be the denominator of this quotient produces

$$u = \sec x + \tan x \quad \Longrightarrow \quad u' = \sec x \tan x + \sec^2 x.$$

- So, you can conclude that

$$\int \sec x \, \mathrm{d}x = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, \mathrm{d}x$$
$$\int \frac{u'}{u} \, \mathrm{d}x = \ln |u| + C = \ln |\sec x + \tan x| + C. \qquad \blacksquare$$

Table 1: Integrals of the six basic trigonometric functions

$\int \sin u \, \mathrm{d}u = -\cos u + C$	$\int \cos u \, \mathrm{d}u = \sin u + C$
$\int \tan u \, \mathrm{d}u = -\ln\lvert \cos u\rvert + C$	$\int \cot u \, \mathrm{d}u = \ln\lvert \sin u\rvert + C$
$\int \sec u \, \mathrm{d}u = \ln\lvert \sec u + \tan u\rvert + C$	$\int \csc u \, \mathrm{d}u = -\ln\lvert \csc u + \cot u\rvert + C$

Example 10 (Integrating trigonometric functions)

Evaluate $\int_0^{\pi/4} \sqrt{1+\tan^2 x}\,\mathrm{d}x$.

Using $1+\tan^2 x = \sec^2 x$, you can write

$$\int_0^{\pi/4} \sqrt{1+\tan^2 x}\,\mathrm{d}x = \int_0^{\pi/4} \sqrt{\sec^2 x}\,\mathrm{d}x = \int_0^{\pi/4} \sec x\,\mathrm{d}x$$

$$= \ln\lvert \sec x + \tan x\rvert\Big|_0^{\pi/4} = \ln\left(\sqrt{2}+1\right) - \ln 1 \approx 0.881.$$

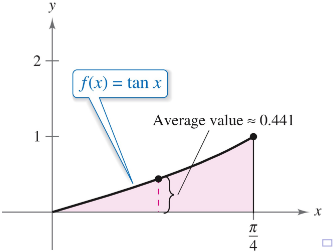
∎

Example 11 (Finding an average value)

Find the average value of $f(x) = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$.

$$\begin{aligned}
\text{Average value} &= \frac{1}{(\pi/4) - 0} \int_0^{\pi/4} \tan x \, dx \\
&= \frac{4}{\pi} \int_0^{\pi/4} \tan x \, dx = \frac{4}{\pi} \left[-\ln|\cos x| \right]_0^{\pi/4} \\
&= -\frac{4}{\pi} \left[\ln\left(\frac{\sqrt{2}}{2}\right) - \ln(1) \right] = -\frac{4}{\pi} \ln\left(\frac{\sqrt{2}}{2}\right) \approx 0.441
\end{aligned}$$

The average value is about 0.441, as shown in the following Figure. ■



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Inverse functions

- The function $f(x) = x + 3$ from $A = \{1, 2, 3, 4\}$ to $B = \{4, 5, 6, 7\}$ can be written as

$$f\colon \{(1, 4),\ (2, 5),\ (3, 6),\ (4, 7)\}.$$

- By interchanging the first and second coordinates of each ordered pair, you can form the inverse function of f. This function is denoted by f^{-1}. It is a function from B to A, and can be written as

$$f^{-1}\colon \{(4, 1),\ (5, 2),\ (6, 3),\ (7, 4)\}.$$

- The domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 8.
- The functions f and f^{-1} have the effect of "undoing" each other. That is, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$

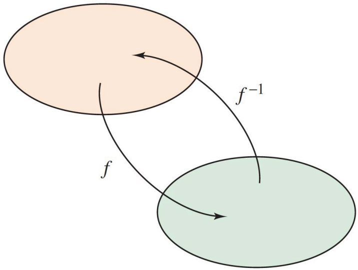


Figure 8: Domain of f = range of f^{-1}, Domain of f^{-1} = range of f.

Definition 5.3 (Inverse function)

A function g is the inverse function of the function f if $f(g(x)) = x$ for each x in the domain of g and $g(f(x)) = x$ for each x in the domain of f. The function g is denoted by f^{-1} (read "f inverse").

Here are some important observations about inverse functions.

1. If g is the inverse function of f, then f is the inverse function of g.
2. The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.
3. A function need not have an inverse function, but if it does, the inverse function is unique.

- You can think of f^{-1} as undoing what has been done by f.
- For example, subtraction can be used to undo addition, and division can be used to undo multiplication.
- Use the definition of an inverse function to check the following.
- $f(x) = x + c$ and $f^{-1}(x) = x - c$ are inverse functions of each other.
- $f(x) = cx$ and $f^{-1}(x) = \frac{x}{c}$, $c \neq 0$, are inverse functions of each other.

Example 1 (Verifying inverse functions)

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

- Because the domains and ranges of both f and g consist of all real numbers, you can conclude that both composite functions exist for all x.
- The composition of f with g is given by

$$f(g(x)) = 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 = 2\left(\frac{x+1}{2}\right) - 1 = x + 1 - 1 = x.$$

- The composition of g with f is given by

$$g(f(x)) = \sqrt[3]{\frac{(2x^3-1)+1}{2}} = \sqrt[3]{\frac{2x^3}{2}} = \sqrt[3]{x^3} = x.$$

- Because $f(g(x)) = x$ and $g(f(x)) = x$, you can conclude that f and g are inverse functions of each other (see Figure 9). ■

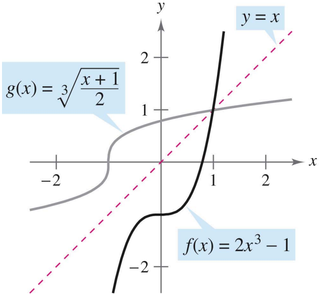


Figure 9: $f(x) = 2x^3 - 1$ and $g(x) = \sqrt[3]{\frac{x+1}{2}}$ are inverse functions of each other.

- In Figure 9, the graphs of f and $g = f^{-1}$ appear to be mirror images of each other with respect to the line $y = x$.
- The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.
- The idea of a reflection of the graph of f in the line $y = x$ is generalized in the following theorem.

Theorem 5.6 (Reflective property of inverse functions)

The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

- If (a, b) is on the graph of f, then $f(a) = b$ and you can write
$$f^{-1}(b) = f^{-1}(f(a)) = a.$$
- So, (b, a) is on the graph of f^{-1}, as shown in Figure 10.
- A similar argument will prove the theorem in the other direction. $\square$

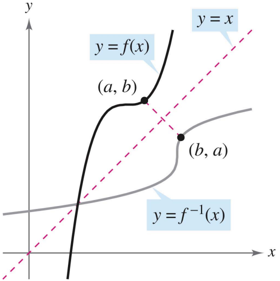


Figure 10: The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

Existence of an inverse function

- Not every function has an inverse function, and Theorem 5.6 suggests a graphical test for those that do—the Horizontal Line Test for an inverse function.
- This test states that a function f has an inverse function if and only if every horizontal line intersects the graph of f at most once (see Figure 11).

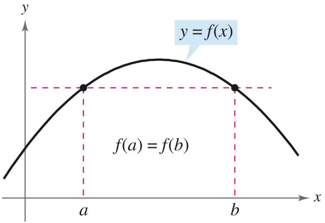


Figure 11: If a horizontal line intersects the graph of f twice, then f is not one-to-one.

- The following theorem formally states why the Horizontal Line Test is valid.

Theorem 5.7 (The existence of an inverse function)

1. *A function has an inverse function if and only if it is one-to-one.*
2. *If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.*

- To prove the second part of the theorem, recall from Section P.3 that f is one-to-one if for x_1 and x_2 in its domain

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2).$$

- Now, choose x_1 and x_2 in the domain of f.
- If $x_1 \neq x_2$, then, because f is strictly monotonic, it follows that either

$$f(x_1) < f(x_2) \quad \text{or} \quad f(x_1) > f(x_2).$$

- In either case, $f(x_1) \neq f(x_2)$. So, f is one-to-one on the interval.
- The proof of the first part of the theorem is left as an exercise (see Exercise 95).

Example 2 (The existence of an inverse function)

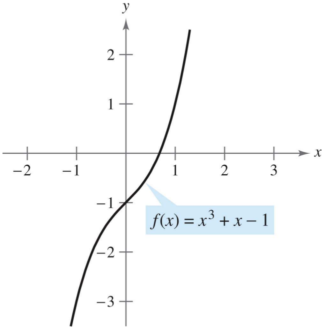
Which of the functions has an inverse function?
a. $f(x) = x^3 + x - 1$ **b.** $f(x) = x^3 - x + 1$

a. From the graph of f shown in Figure 12, it appears that f is increasing over its entire domain.

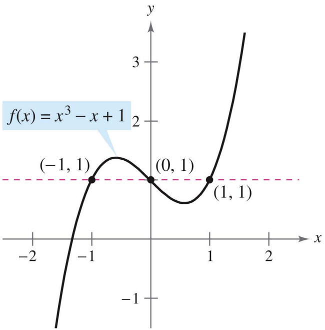
- To verify this, note that the derivative, $f'(x) = 3x^2 + 1$, is positive for all real values of x.
- So, f is strictly monotonic and it must have an inverse function.

b. From the graph of f shown in Figure 12, you can see that the function does not pass the horizontal line test.

- In other words, it is not one-to-one. For instance, f has the same value when $x = -1$, 0, and 1.
- $f(-1) = f(1) = f(0) = 1$ Not one-to-one
- So, by Theorem 5.7, f does not have an inverse function. ∎



(a) Because $f(x) = x^3 + x - 1$ is increasing over its entire domain, it has an inverse function.



(b) Because $f(x) = x^3 - x + 1$ is not one-to-one, it does not have an inverse function.

Figure 12: The existence of an inverse function.

- The following guidelines suggest a procedure for finding an inverse function.

Guidelines for finding an inverse function

1. Use Theorem 5.7 to determine whether the function given by $y = f(x)$ has an inverse function.
2. Solve for x as a function of $y : x = g(y) = f^{-1}(y)$.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} as the range of f.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Example 3 (Finding an inverse function)

Find the inverse function of $f(x) = \sqrt{2x - 3}$.

- From the graph of f in Figure 13, it appears that f is increasing over its entire domain, $\left[\frac{3}{2}, \infty\right)$.
- To verify this, note that $f'(x) = \frac{1}{\sqrt{2x-3}}$ is positive on the domain of f. So, f is strictly monotonic and it must have an inverse function.
- To find an equation for the inverse function, let $y = f(x)$ and solve for x in terms of y.

$$\sqrt{2x-3} = y \qquad 2x - 3 = y^2 \qquad x = \frac{y^2+3}{2}$$

$$y = \frac{x^2+3}{2} \qquad f^{-1}(x) = \frac{x^2+3}{2}$$

- The domain of f^{-1} is the range of f which is $[0, \infty)$.
- You can verify this result as shown.

$$f(f^{-1}(x)) = \sqrt{2\left(\frac{x^2+3}{2}\right) - 3} = \sqrt{x^2} = x, \quad x \geq 0$$

$$f^{-1}(f(x)) = \frac{(\sqrt{2x-3})^2+3}{2} = \frac{2x-3+3}{2} = x, \quad x \geq \frac{3}{2} \qquad \blacksquare$$

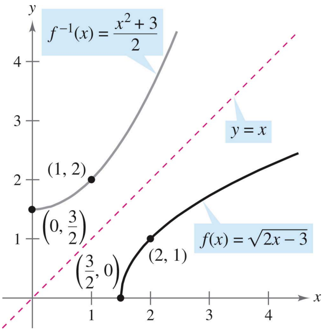


Figure 13: The domain of $f^{-1}(x) = \frac{x^2+3}{2}$, $[0, \infty)$ is the range of $f(x) = \sqrt{2x-3}$.

- Suppose you are given a function that is not one-to-one on its domain.
- By restricting the domain to an interval on which the function is strictly monotonic, you can conclude that the new function is one-to-one on the restricted domain.

Example 4 (Testing whether a function is one-to-one)

Show that the sine function

$$f(x) = \sin x$$

is not one-to-one on the entire real line. Then show that $[-\pi/2, \pi/2]$ is the largest interval, centered at the origin, on which f is strictly monotonic.

- It is clear that f is not one-to-one, because many different x-values yield the same y-value.
- For instance,

$$\sin(0) = 0 = \sin(\pi).$$

- Moreover, f is increasing on the open interval $(-\pi/2, \pi/2)$, because its derivative

$$f'(x) = \cos x$$

is positive there.

- Finally, because the left and right endpoints correspond to relative extrema of the sine function, you can conclude that f is increasing on the closed interval $[-\pi/2, \pi/2]$ and that on any larger interval the function is not strictly monotonic (see Figure 14). ■

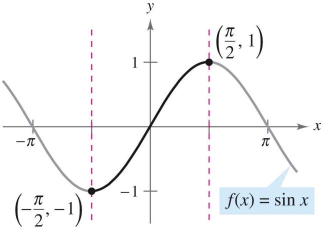


Figure 14: $f(x) = \sin x$ is one-to-one on the interval $[-\pi/2, \pi/2]$.

Derivative of an inverse function

The next two theorems discuss the derivative of an inverse function.

Theorem 5.8 (Continuity and differentiability of inverse functions)

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. *If f is continuous on its domain, then f^{-1} is continuous on its domain.*
2. *If f is increasing on its domain, then f^{-1} is increasing on its domain.*
3. *If f is decreasing on its domain, then f^{-1} is decreasing on its domain.*
4. *If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.*

Theorem 5.9 (The derivative of an inverse function)

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

Example 5 (Evaluating the derivative of an inverse function)

Let $f(x) = \frac{1}{4}x^3 + x - 1$.

a. What is the value of $f^{-1}(x)$ when $x = 3$?

b. What is the value of $(f^{-1})'(x)$ when $x = 3$?

- Notice that f is one-to-one and therefore has an inverse function.

a. Because $f(x) = 3$ when $x = 2$, you know that $f^{-1}(3) = 2$.

b. Because the function f is differentiable and has an inverse function, you can apply Theorem 5.9 (with $g = f^{-1}$) to write

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(2)}.$$

- Moreover, using $f'(x) = \frac{3}{4}x^2 + 1$, you can conclude that

$$(f^{-1})'(3) = \frac{1}{f'(2)} = \frac{1}{\frac{3}{4}(2^2) + 1} = \frac{1}{4}.$$

$\blacksquare$

- In Example 5, note that at the point $(2, 3)$ the slope of the graph of f is 4 and at the point $(3, 2)$ the slope of the graph of f^{-1} is $\frac{1}{4}$. (see Figure 15).
- This reciprocal relationship can be written as shown below.

- If $y = g(x) = f^{-1}(x)$, then $f(y) = x$ and $f'(y) = \frac{\mathrm{d}x}{\mathrm{d}y}$. Theorem 5.9 says that

$$g'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{f'(g(x))} = \frac{1}{f'(y)} = \frac{1}{(\mathrm{d}x/\mathrm{d}y)}.$$

- So, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}$.

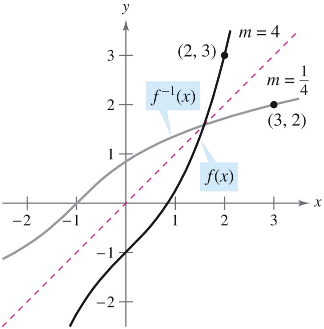


Figure 15: The graphs of the inverse functions f and f^{-1} have reciprocal slopes at points (a, b) and (b, a).

Example 6 (Graphs of inverse functions have reciprocal slopes)

Let $f(x) = x^2$ (for $x \geq 0$) and let $f^{-1}(x) = \sqrt{x}$. Show that the slopes of the graphs of f and f^{-1} are reciprocals at each of the following points.

a. $(2, 4)$ and $(4, 2)$

b. $(3, 9)$ and $(9, 3)$

- The derivative of f and f^{-1} are given by

$$f'(x) = 2x \quad \text{and} \quad (f^{-1})'(x) = \frac{1}{2\sqrt{x}}.$$

a. At $(2, 4)$, the slope of the graph of f is $f'(2) = 2(2) = 4$.

- At $(4, 2)$, the slope of the graph of f^{-1} is

$$(f^{-1})'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = \frac{1}{4}.$$

- At $(3, 9)$, the slope of the graph of f is $f'(3) = 2(3) = 6$.
- At $(9, 3)$, the slope of the graph of f^{-1} is

$$(f^{-1})'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}.$$

- So, in both cases, the slopes are reciprocals, as shown in Figure 16. ■

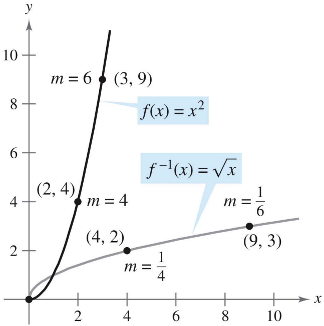


Figure 16: At $(0,0)$, the derivative of $f(x) = x^2$ is 0, and the derivative of $f^{-1}(x) = \sqrt{x}$ does not exist.

Table of Contents

The natural exponential function

- The function $f(x) = \ln x$ is increasing on its entire domain, and therefore it has an inverse function f^{-1}.
- The domain of f^{-1} is the set of all reals, and the range is the set of positive reals, as shown in Figure 17.

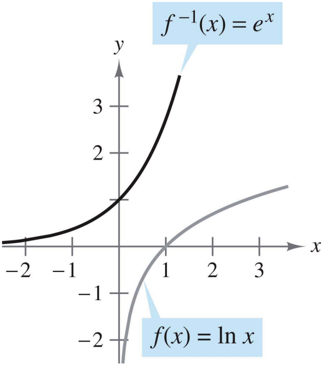


Figure 17: The inverse function of the natural logarithmic function is the natural exponential function.

- So, for any real number x,

$$f(f^{-1}(x)) = \ln[f^{-1}(x)] = x. \qquad x \text{ is any real number}$$

- If x happens to be rational, then

$$\ln(e^x) = x \ln e = x(1) = x. \qquad x \text{ is a rational number}$$

- Because the natural logarithmic function is one-to-one, you can conclude that $f^{-1}(x)$ and e^x agree for rational values of x. The following definition extends the meaning of e^x to include all real values of x.

Definition 5.4 (The natural exponential function)

The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the natural exponential function and is denoted by

$$f^{-1}(x) = e^x.$$

That is,

$$y = e^x \quad \text{if and only if} \quad x = \ln y.$$

The symbol e was first used by mathematician Leonhard Euler to represent the base of natural logarithms in a letter to another mathematician, Christian Goldbach, in 1731.

- The inverse relationship between the natural logarithmic function and the natural exponential function can be summarized as follows.

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x \qquad \text{Inverse relationship}$$

Example 1 (Solving an exponential equation)

Solve $7 = e^{x+1}$.

- You can convert from exponential form to logarithmic form by taking the natural logarithm of each side of the equation.

$$7 = e^{x+1} \quad \ln 7 = \ln(e^{x+1}) \quad \ln 7 = x + 1 \quad -1 + \ln 7 = x \quad 0.946 \approx x$$

■

Example 2 (Solving a logarithmic equation (exponentiate))

Solve $\ln(2x-3)=5$.

$$\ln(2x-3)=5 \implies e^{\ln(2x-3)}=e^5 \implies 2x-3=e^5$$
$$\implies x=\frac{1}{2}(e^5+3) \implies x\approx 75.707 \qquad \blacksquare$$

Theorem 5.10 (Operations with exponential functions)

Let a and b be any real numbers.

1. $e^a e^b = e^{a+b}$
2. $\frac{e^a}{e^b} = e^{a-b}$

- To prove Property 1, you can write

$$\ln(e^a e^b)=\ln(e^a)+\ln(e^b)=a+b=\ln(e^{a+b}).$$

- Because the natural logarithmic function is one-to-one, you can conclude that
$$e^a e^b = e^{a+b}.$$
- To prove Property 2, you can write
$$\ln\left(\frac{e^a}{e^b}\right) = \ln e^a - \ln e^b = a - b = \ln(e^{a-b}).$$
- Because the natural logarithmic function is one-to-one, you can conclude that
$$\frac{e^a}{e^b} = e^{a-b}.$$

$\square$

- An inverse function f^{-1} shares many properties with f.
- So, the natural exponential function inherits the following properties from the natural logarithmic function (see Figure 18).

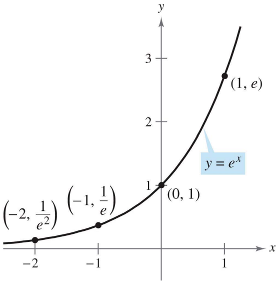


Figure 18: The natural exponential function is increasing, and its graph is concave upward.

Properties of the natural exponential function

1. The domain of $f(x) = e^x$ is $(-\infty, \infty)$, and the range is $(0, \infty)$.
2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.
3. The graph of $f(x) = e^x$ is concave upward on its entire domain.
4. $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^x = \infty$.

Derivatives of exponential functions

- One of the most intriguing (and useful) characteristics of the natural exponential function is that it is its own derivative.
- In other words, it is a solution to the differential equation $y' = y$. This result is stated in the next theorem.

Theorem 5.11 (Derivatives of the natural exponential function)

Let u be a differentiable function of x.

1. $\frac{\mathrm{d}}{\mathrm{d}x}[e^x] = e^x$
2. $\frac{\mathrm{d}}{\mathrm{d}x}[e^u] = e^u \frac{\mathrm{d}u}{\mathrm{d}x}$

- To prove Property 1, use the fact that $\ln e^x = x$, and differentiate each side of the equation.

$$\ln e^x = x \quad \frac{\mathrm{d}}{\mathrm{d}x}[\ln e^x] = \frac{\mathrm{d}}{\mathrm{d}x}[x] \quad \frac{1}{e^x}\frac{\mathrm{d}}{\mathrm{d}x}[e^x] = 1 \quad \frac{\mathrm{d}}{\mathrm{d}x}[e^x] = e^x$$

- The derivative of e^u follows from the Chain Rule. □

Example 3 (Differentiating exponential functions)

a. $\frac{d}{dx}[e^{2x-1}] = e^u \frac{du}{dx} = 2e^{2x-1} \qquad u = 2x - 1$

b. $\frac{d}{dx}[e^{-3/x}] = e^u \frac{du}{dx} = \left(\frac{3}{x^2}\right) e^{-3/x} = \frac{3e^{-3/x}}{x^2} \qquad u = -\frac{3}{x}$

c. $\frac{d}{dx}[x^2 e^x] = x^2(e^x) + e^x(2x) = xe^x(x+2)$
Product rule and Theorem 5.11

d. $\frac{d}{dx}\left[\frac{e^{3x}}{e^x+1}\right] = \frac{(e^x+1)(3e^{3x}) - e^{3x}(e^x)}{(e^x+1)^2} = \frac{3e^{4x}+3e^{3x}-e^{4x}}{(e^x+1)^2} = \frac{e^{3x}(2e^x+3)}{(e^x+1)^2}$ ■

Example 4 (Locating relative extrema)

Find the relative extrema of $f(x) = xe^x$.

The derivative of f is given by

$$f'(x) = x(e^x) + e^x(1) = e^x(x+1)$$

- Because e^x is never 0, the derivative is 0 only when $x = -1$.

- Moreover, by the First Derivative Test, you can determine that this corresponds to a relative minimum, as shown in Figure 19.
- Because the derivative $f'(x) = e^x(x+1)$ is defined for all x, there are no other critical points. ■

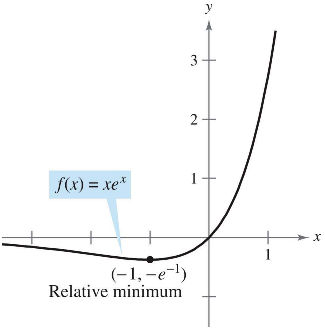


Figure 19: The derivative of f changes from negative to positive at $x = -1$.

Example 5 (Finding an equation of a tangent line)

Find an equation of the tangent line to the graph of $f(x) = 2 + e^{1-x}$ at the point $(1, 3)$.

- Begin by finding $f'(x)$.

$$f(x) = 2 + e^{1-x} \qquad f'(x) = e^{1-x}(-1) = -e^{1-x}$$

- To find the slope of the tangent line at $(1, 3)$, evaluate $f'(1)$.

$$f'(1) = -e^{1-1} = -e^0 = -1$$

- Now, using the point-slope form of the equation of a line, you can write

$$y - y_1 = m(x - x_1) \qquad y - 3 = -1(x - 1) \qquad y = -x + 4.$$

- The graph of f and its tangent line at $(1, 3)$ are shown in Figure 20. ■

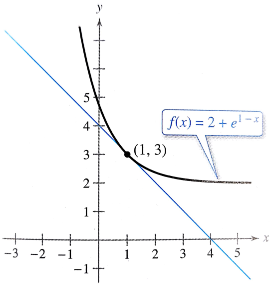


Figure 20: The tangent line of $f(x) = 2 + e^{1-x}$ at $(1, 3)$.

Example 6 (The standard normal probability density function)

Show that the standard normal probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

has points of inflection when $x = \pm 1$.

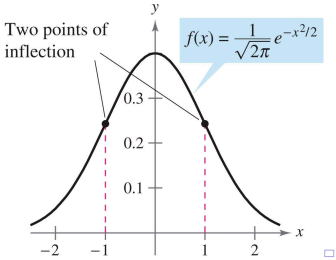
- To locate possible points of inflection, find the x-values for which the second derivative is 0.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$f'(x) = \frac{1}{\sqrt{2\pi}} (-x)e^{-x^2/2}$$

$$f''(x) = \frac{1}{\sqrt{2\pi}} [(-x)(-x)e^{-x^2/2} + (-1)e^{-x^2/2}] = \frac{1}{\sqrt{2\pi}} (e^{-x^2/2})(x^2 - 1)$$

- So, $f''(x) = 0$ when $x = \pm 1$, and you can apply the techniques of Chapter 3 to conclude that these values yield the two points of inflection shown in the following Figure. ■



Theorem 5.12 (Integration rules for exponential functions)

Let u be a differentiable function of x.
1. $\int e^x \,\mathrm{d}x = e^x + C$ *2.* $\int e^u \,\mathrm{d}u = e^u + C$

Example 7 (Integrating exponential functions)

Find $\int e^{3x+1} \,\mathrm{d}x$.

If you let $u = 3x + 1$, then $\mathrm{d}u = 3\,\mathrm{d}x$.

$$\int e^{3x+1} \,\mathrm{d}x = \frac{1}{3} \int e^{3x+1}(3) \,\mathrm{d}x = \frac{1}{3} \int e^u \,\mathrm{d}u = \frac{1}{3} e^u + C = \frac{e^{3x+1}}{3} + C$$

∎

Example 8 (Integrating exponential functions)

Find $\int 5xe^{-x^2}\,\mathrm{d}x$.

If you let $u = -x^2$, then $\mathrm{d}u = -2x\,\mathrm{d}x$ or $x\,\mathrm{d}x = -\,\mathrm{d}u/2$.

$$\int 5xe^{-x^2}\,\mathrm{d}x = \int 5e^{-x^2}(x\,\mathrm{d}x) = \int 5e^{u}\left(-\frac{\mathrm{d}u}{2}\right)$$
$$= -\frac{5}{2}\int e^{u}\,\mathrm{d}u = -\frac{5}{2}e^{u} + C = -\frac{5}{2}e^{-x^2} + C \qquad \blacksquare$$

Example 9 (Integrating exponential functions)

a. $\int \frac{e^{1/x}}{x^2}\,\mathrm{d}x = -\int e^{1/x}\left(-\frac{1}{x^2}\right)\,\mathrm{d}x = -e^{1/x} + C \qquad u = \frac{1}{x}$

b. $\int \sin x\, e^{\cos x}\,\mathrm{d}x = -\int e^{\cos x}(-\sin x)\,\mathrm{d}x = -e^{\cos x} + C \qquad u = \cos x$ $\blacksquare$

Example 10 (Finding areas bounded by exponential functions)

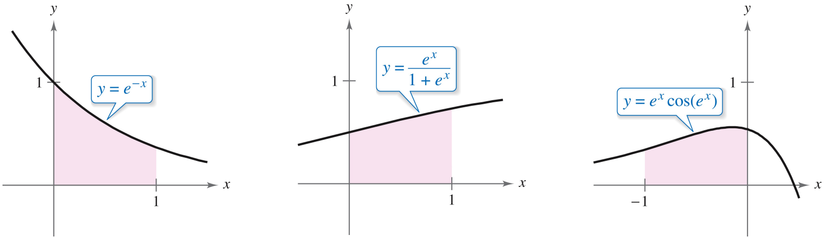
a. $\int_0^1 e^{-x}\,\mathrm{d}x$ **b.** $\int_0^1 \frac{e^x}{1+e^x}\,\mathrm{d}x$ **c.** $\int_{-1}^0 [e^x\cos(e^x)]\,\mathrm{d}x$

a. $\int_0^1 e^{-x}\,\mathrm{d}x = [-e^{-x}]_0^1 = -e^{-1} - (-1) = 1 - \frac{1}{e} \approx 0.632$

b. $\int_0^1 \frac{e^x}{1+e^x}\,\mathrm{d}x = [\ln(1+e^x)]_0^1 = \ln(1+e) - \ln 2 \approx 0.620$

c. $\int_{-1}^0 [e^x\cos(e^x)]\,\mathrm{d}x = [\sin(e^x)]_{-1}^0 = \sin 1 - \sin(e^{-1}) \approx 0.482$

■



(a) $y = e^{-x}$ (b) $y = \frac{e^x}{1+e^x}$ (c) $y = e^x\cos(e^x)$

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- The base of the natural exponential function is e. This "natural" base can be used to assign a meaning to a general base a.

Definition 5.5 (Exponential function to base a)

If a is a positive real number ($a \neq 1$) and x is any real number, then the exponential function to the base a is denoted by a^x and is defined by

$$a^x = e^{(\ln a)x}.$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

- These functions obey the usual laws of exponents. For instance, here are some familiar properties.
  1. $a^0 = 1$
  2. $a^x a^y = a^{x+y}$
  3. $\frac{a^x}{a^y} = a^{x-y}$
  4. $(a^x)^y = a^{xy}$

- When modeling the half-life of a radioactive sample, it is convenient to use $\frac{1}{2}$ as the base of the exponential model. (Half-life is the number of years required for half of the atoms in a sample of radioactive material to decay.)

Definition 5.6 (Logarithmic function to base a)

If a is a positive real number ($a \neq 1$) and x is any positive real number, then the logarithmic function to the base a is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x.$$

- Logarithmic functions to the base a have properties similar to those of the natural logarithmic function. $a > 0$, $a \neq 1$, x, $y > 0$
  1. $\log_a 1 = 0$ Log of 1
  2. $\log_a xy = \log_a x + \log_a y$ Log of a product
  3. $\log_a x^n = n \log_a x$ Log of a power
  4. $\log_a \frac{x}{y} = \log_a x - \log_a y$ Log of a quotient

- From the definitions of the exponential and logarithmic functions to the base a, it follows that $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions of each other.

Properties of inverse functions

1. $y = a^x$ if and only if $x = \log_a y$.
2. $a^{\log_a x} = x$, for $x > 0$.
3. $\log_a a^x = x$, for all x.

- The logarithmic function to the base 10 is called the common logarithmic function. So, for common logarithms, $y = 10^x$ if and only if $x = \log_{10} y$.

Example 2 (Bases other than e)

Solve for x in each equation. **a.** $3^x = \frac{1}{81}$ **b.** $\log_2 x = -4$

a. To solve this equation, you can apply the logarithmic function to the base 3 to each side of the equation.

$$3^x = \frac{1}{81} \qquad \log_3 3^x = \log_3 \frac{1}{81} \qquad x = \log_3 3^{-4} \qquad x = -4$$

b. To solve this equation, you can apply the exponential function to the base 2 to each side of the equation.

$$\log_2 x = -4 \qquad 2^{\log_2 x} = 2^{-4} \qquad x = \frac{1}{2^4} \qquad x = \frac{1}{16} \qquad \blacksquare$$

Differentiation and integration

- To differentiate exponential and logarithmic functions to other bases, you have three options:
  (1) use the definitions of a^x and $\log_a x$ and differentiate using the rules for the natural exponential and logarithmic functions,
  (2) use logarithmic differentiation, or
  (3) use the following differentiation rules for bases other than e.

Theorem 5.13 (Derivatives for bases other than e)

Let a be a positive real number ($a \neq 1$) and let u be a differentiable function of x.

1. $\frac{\mathrm{d}}{\mathrm{d}x}[a^x] = (\ln a)a^x$
2. $\frac{\mathrm{d}}{\mathrm{d}x}[a^u] = (\ln a)a^u \frac{\mathrm{d}u}{\mathrm{d}x}$
3. $\frac{\mathrm{d}}{\mathrm{d}x}[\log_a x] = \frac{1}{(\ln a)x}$
4. $\frac{\mathrm{d}}{\mathrm{d}x}[\log_a u] = \frac{1}{(\ln a)u} \frac{\mathrm{d}u}{\mathrm{d}x}$

- By definition, $a^x = e^{(\ln a)x}$.
- So, you can prove the first rule by letting $u = (\ln a)x$ and differentiating with base e to obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\,[a^x] = \frac{\mathrm{d}}{\mathrm{d}x}\left[e^{(\ln a)x}\right] = e^u\,\frac{\mathrm{d}u}{\mathrm{d}x} = (\ln a)a^x.$$

- To prove the third rule, you can write

$$\frac{\mathrm{d}}{\mathrm{d}x}\,[\log_a x] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\ln a}\ln x\right] = \frac{1}{\ln a}\left(\frac{1}{x}\right) = \frac{1}{(\ln a)x}.$$

- The second and fourth rules are simply the Chain Rule versions of the first and third rules. $\square$

Example 3 (Differentiating functions to other bases)

Find the derivative of each function.

a. $y = 2^x$ **b.** $y = 2^{3x}$ **c.** $y = \log_{10} \cos x$ **d.** $y = \log_3 \frac{\sqrt{x}}{x+5}$

a. $y' = \frac{d}{dx}[2^x] = (\ln 2)2^x$

b. $y' = \frac{d}{dx}[2^{3x}] = (\ln 2)2^{3x}(3) = (3\ln 2)2^{3x}$
Try writing 2^{3x} as 8^x and differentiating to see that you obtain the same result.

c. $y' = \frac{d}{dx}[\log_{10} \cos x] = \frac{-\sin x}{(\ln 10)\cos x} = -\frac{\tan x}{\ln 10}$

d. Before differentiating, rewrite the function using logarithmic properties.

$$y = \log_3 \frac{\sqrt{x}}{x+5} = \frac{1}{2}\log_3 x - \log_3(x+5)$$

- Next, apply Theorem 5.13 toe differentiate the function.

$$\begin{aligned} y' &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\log_3 x - \log_3(x+5)\right] \\ &= \frac{1}{2(\ln 3)x} - \frac{1}{(\ln 3)(x+5)} = \frac{5-x}{2(\ln 3)x(x+5)} \end{aligned}$$

■

- Occasionally, an integrand involves an exponential function to a base other than e. When this occurs, there are two options:
  1. convert to base e using the formula $a^x = e^{(\ln a)x}$ and then integrate, or
  2. integrate directly, using the integration formula

$$\int a^x \,\mathrm{d}x = \left(\frac{1}{\ln a}\right) a^x + C.$$

Example 4 (Integrating an exponential function to another base)

Find $\int 2^x \,\mathrm{d}x$.

$$\int 2^x \,\mathrm{d}x = \int e^{\ln(2^x)} \,\mathrm{d}x = \int e^{x \ln 2} \,\mathrm{d}x = \frac{1}{\ln 2} 2^x + C$$

■

Theorem 5.14 (The Power Rule for real exponents)

Let n be any real number and let u be a differentiable function of x.

1. $\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1}$
2. $\frac{\mathrm{d}}{\mathrm{d}x}[u^n] = nu^{n-1}\frac{\mathrm{d}u}{\mathrm{d}x}$

Example 5 (Comparing variables and constants)

a. $\frac{\mathrm{d}}{\mathrm{d}x}[e^e] = 0$ Constant Rule

b. $\frac{\mathrm{d}}{\mathrm{d}x}[e^x] = e^x$ Exponential Rule

c. $\frac{\mathrm{d}}{\mathrm{d}x}[x^e] = ex^{e-1}$ Power Rule

d. $y = x^x$ Logarithmic differentiation

d.

$$\ln y = \ln x^x \qquad \ln y = x \ln x$$

$$\frac{y'}{y} = x\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x \qquad y' = y(1 + \ln x) = x^x(1 + \ln x)$$

Applications of exponential functions

- Suppose P dollars is deposited in an account at an annual interest rate r (in decimal form). If interest accumulates in the account, what is the balance in the account at the end of 1 year? The answer depends on the number of times n the interest is compounded according to the formula

$$A = P\left(1 + \frac{r}{n}\right)^n.$$

- For instance, the result for a deposit of \$1000 at 8% interest compounded n times a year is shown in the table.

n	A
1	\$1080.00
2	\$1081.60
4	\$1082.33
12	\$1083.00
365	\$1083.28

- As n increases, the balance A approaches a limit. To develop this limit, use the following theorem.

Theorem 5.15 (A limit involving e)

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = e$$

- To test the reasonableness of this theorem, try evaluating $[(x+1)/x]^x$ for several values of x, as shown in the table.

x	$\left(\frac{x+1}{x}\right)^x$
10	2.59374
100	2.70481
$1,000$	2.71692
$10,000$	2.71815
$100,000$	2.71827
$1,000,000$	2.71828

- Now, let's take another look at the formula for the balance A in an account in which the interest is compounded n times per year.
- By taking the limit as n approaches infinity, you obtain

$$
\begin{aligned}
A &= \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^n = P\lim_{n\to\infty}\left[\left(1+\frac{1}{n/r}\right)^{n/r}\right]^r \\
&= P\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]^r = Pe^r.
\end{aligned}
$$

- This limit produces the balance after 1 year of continuous compounding. So, for a deposit of 1000 at 8% interest compounded continuously, the balance at the end of 1 year would be

$$A = 1000e^{0.08} \approx \$1083.29.$$

Summary of compound interest formulas Let $P =$ amount of deposit, $t =$ number of years, $A =$ balance after t years, $r =$ annual interest rate (decimal form), and $n =$ number of compoundings per year.

1. Compounded n times per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$
2. Compounded continuously: $A = Pe^{rt}$

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Indeterminate forms

- The forms $0/0$ and ∞/∞ are called indeterminate because they do not guarantee that a limit exists, nor do they indicate what the limit is, if one does exist.
- When you encountered one of these indeterminate forms earlier in the text, you attempted to rewrite the expression by using various algebraic techniques.

Indeterminate forms	*Limit*	*Algebraic technique*
$\frac{0}{0}$	$\lim_{x\to-1}\frac{2x^2-2}{x+1}$ $=\lim_{x\to-1}2(x-1)=-4$	Divide numerator and denominator by $(x+1)$.
$\frac{\infty}{\infty}$	$\lim_{x\to\infty}\frac{3x^2-1}{2x^2+1}$ $=\lim_{x\to\infty}\frac{3-(1/x^2)}{2+(1/x^2)}=\frac{3}{2}$	Divide numerator and denominator by x^2.

- You can extend these algebraic techniques to find limits of transcendental functions. For instance, the limit

$$\lim_{x\to 0}\frac{e^{2x}-1}{e^x-1}$$

produces the indeterminate form $0/0$. Factoring and then dividing produces

$$\lim_{x\to 0}\frac{e^{2x}-1}{e^x-1}=\lim_{x\to 0}\frac{(e^x+1)(e^x-1)}{e^x-1}=\lim_{x\to 0}(e^x+1)=2.$$

However, not all indeterminate forms can be evaluated by algebraic manipulation. This is often true when both algebraic and transcendental functions are involved. For instance, the limit

$$\lim_{x\to 0}\frac{e^{2x}-1}{x}$$

produces the indeterminate form $0/0$.Rewriting the expression to obtain

$$\lim_{x\to 0}\left(\frac{e^{2x}}{x}-\frac{1}{x}\right)$$

merely produces another indeterminate form, $\infty-\infty$.You could use technology to estimate the limit, as shown in the table and in Figure 23. From the table and the graph, the limit appears to be 2.

-

x	-1	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	1
$\frac{e^{2x}-1}{x}$	0.865	1.813	1.980	1.998	?	2.002	2.020	2.214	6.389

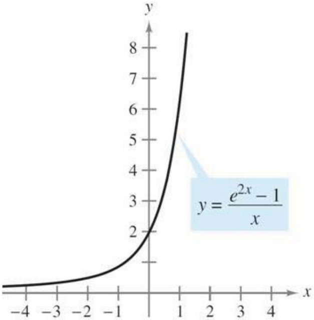

Figure 23: The limit as x approaches 0 appears to be 2.

- To find the limit illustrated in Figure 23, you can use a theorem called L'Hôpital's Rule. This theorem states that under certain conditions the limit of the quotient $f(x)/g(x)$ is determined by the limit of the quotient of the derivatives $\frac{f'(x)}{g'(x)}$.
- To prove this theorem, you can use a more general result called the Extended Mean Value Theorem.

Theorem 5.16 (The Extended Mean Value Theorem)

If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Theorem 5.17 (L'Hôpital's Rule)

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as x approaches c produces anyone of the indeterminate forms ∞/∞,$(-\infty)/\infty$, $\infty/(-\infty)$ or $(-\infty)/(-\infty)$.

Example 1 (Indeterminate form 0/0)

Evaluate $\lim_{x \to 0} \frac{e^{2x}-1}{x}$.

- Because direct substitution results in the indeterminate form 0/0.

$$\lim_{x \to 0} (e^{2x} - 1) = 0$$

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x}$$

$$\lim_{x \to 0} x = 0$$

- You can apply L'Hôpital's Rule, as shown below.

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}\left[e^{2x} - 1\right]}{\frac{d}{dx}[x]} = \lim_{x \to 0} \frac{2e^{2x}}{1} = 2. \qquad \blacksquare$$

Example 2 (Indeterminate form $\frac{\infty}{\infty}$)

Evaluate $\lim_{x\to\infty} \frac{\ln x}{x}$.

Because direct substitution results in the indeterminate form ∞/∞, you can apply L'Hôpital's Rule to obtain

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}x}[\ln x]}{\frac{\mathrm{d}}{\mathrm{d}x}[x]} = \lim_{x\to\infty} \frac{1}{x} = 0. \qquad \blacksquare$$

Example 3 (Applying L'Hôpital's Rule more than once)

Evaluate $\lim_{x\to-\infty} \frac{x^2}{e^{-x}}$.

Because direct substitution results in the indeterminate form ∞/∞, you can apply L'Hôpital's Rule to obtain

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0. \qquad \blacksquare$$

Example 4 (Indeterminate form $0 \cdot \infty$)

Evaluate $\lim_{x\to\infty} e^{-x}\sqrt{x}$.

- Because direct substitution produces the indeterminate form $0 \cdot \infty$, you should try to rewrite the limit to fit the form $0/0$ or ∞/∞.
- In this case, you can rewrite the limit to fit the second form.

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^x}.$$

- Now, by L'Hôpital's Rule, you have

$$\lim_{x\to\infty} \frac{\sqrt{x}}{e^x} = \lim_{x\to\infty} \frac{1/(2\sqrt{x})}{e^x} = \lim_{x\to\infty} \frac{1}{2\sqrt{x}e^x} = 0.$$

■

Example 5 (Indeterminate form 1^∞)

Evaluate $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$.

- Because direct substitution yields the indeterminate form 1^∞, you can proceed as follows. To begin, assume that the limit exists and is equal to y.

$$y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

- Taking the natural logarithm of each side produces

$$\ln y = \ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right].$$

- Because the natural logarithmic function is continuous, you can write

$$\begin{aligned}\ln y &= \lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x}\right)\right] = \lim_{x\to\infty}\left(\frac{\ln[1+(1/x)]}{1/x}\right)\\ &= \lim_{x\to\infty}\left(\frac{(-1/x^2)\left\{1/\left[1+(1/x)\right]\right\}}{-1/x^2}\right)\\ &= \lim_{x\to\infty}\frac{1}{1+(1/x)} = 1.\end{aligned}$$

- Now, because you have shown that $\ln y = 1$, you can conclude that $y = e$ and obtain

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

- You can use a graphing utility to confirm this result, as shown in Figure 24. ■

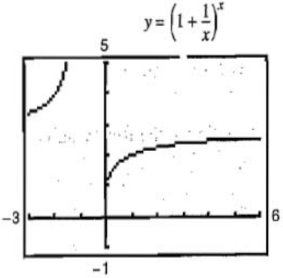

Figure 24: The limit of $[1+(1/x)]^x$ as x approaches infinity is e.

Example 6 (Indeterminate form 0^0)

Find $\lim_{x\to 0^+}(\sin x)^x$.

Because direct substitution produces the indeterminate form 0^0, you can proceed as shown below. To begin, assume that the limit exists and is equal to y.

$$\begin{aligned} y &= \lim_{x\to 0^+}(\sin x)^x \\ \ln y &= \ln\left[\lim_{x\to 0^+}(\sin x)^x\right] = \lim_{x\to 0^+}\left[\ln(\sin x)^x\right] \\ &= \lim_{x\to 0^+}\left[x\ln(\sin x)\right] = \lim_{x\to 0^+}\frac{\ln(\sin x)}{1/x} \\ &= \lim_{x\to 0^+}\frac{\cot x}{-1/x^2} = \lim_{x\to 0^+}\frac{-x^2}{\tan x} = \lim_{x\to 0^+}\frac{-2x}{\sec^2 x} = 0 \end{aligned}$$

Now, because $\ln y = 0$, you can conclude that $y = e^0 = 1$, and it follows that $\lim_{x\to 0^+}(\sin x)^x = 1$. ■

Example 7 (Indeterminate form $\infty - \infty$)

Evaluate $\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$.

Because direct substitution produces the indeterminate form $\infty - \infty$, you should try to rewrite the expression to produce a form to which you can apply L'Hôpital's Rule. In this case, you can combine the two fractions to obtain

$$\begin{aligned}
\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right) &= \lim_{x\to 1^+} \left[\frac{x-1-\ln x}{(x-1)\ln x}\right] = \lim_{x\to 1^+} \left[\frac{1-(1/x)}{(x-1)(1/x)+\ln x}\right] \\
&= \lim_{x\to 1^+} \left[\frac{x-1}{x-1+x\ln x}\right] \\
&= \lim_{x\to 1^+} \left[\frac{1}{1+x(1/x)+\ln x}\right] = \frac{1}{2}. \qquad \blacksquare
\end{aligned}$$

The forms $0/0$, ∞/∞, $\infty - \infty$, $0 \cdot \infty$, 0^0, 1^∞, and ∞^0 have been identified as indeterminate. There are similar forms that you should recognize as "determinate."

$\infty = \infty + \infty$	$\to \infty$	Limit is positive infinity
$-\infty - \infty$	$\to -\infty$	Limit is negative infinity
0^{∞}	$\to 0$	Limit is zero
$0^{-\infty}$	$\to \infty$	Limit is positive infinity

- As a final comment, remember that L'Hôpital's Rule can be applied only to quotients leading to the indeterminate forms $0/0$ and ∞/∞.
- For instance, the following application of L'Hôpital's Rule is incorrect.

$$\lim_{x \to 0} \frac{e^x}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1$$

Incorrect use of L'Hôpital's Rule

- The reason this application is incorrect is that, even though the limit of the denominator is 0, the limit of the numerator is 1, which means that the hypotheses of L'Hôpital's Rule have not been satisfied.

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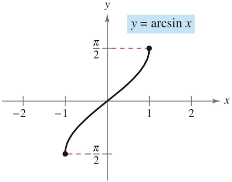
Inverse trigonometric functions

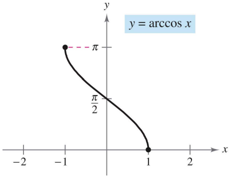
- None of the six basic trigonometric functions has an inverse function.
- This statement is true because all six trigonometric functions are periodic and therefore are not one-to-one.
- In this section you will examine these six functions to see whether their domains can be redefined in such a way that they will have inverse functions on the restricted domains.
- Under suitable restrictions, each of the six trigonometric functions is one-to-one and so has an inverse function, as shown in the following definition.

Function	Domain	Range
$y = \arcsin x$ iff $\sin y = x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \operatorname{arccot} x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \operatorname{arcsec} x$ iff $\sec y = x$	$\lvert x \rvert \geq 1$	$0 \leq y \leq \pi$, $y \neq \frac{\pi}{2}$
$y = \operatorname{arccsc} x$ iff $\csc y = x$	$\lvert x \rvert \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$

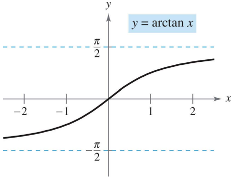
- The graphs of the six inverse trigonometric functions are shown in Figure 25.



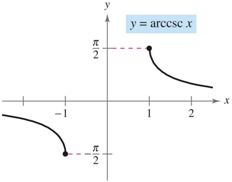
(a) Domain: $[-1, 1]$, Range: $[-\pi/2, \pi/2]$



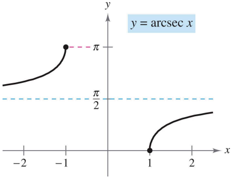
(b) Domain: $[-1, 1]$, Range: $[0, \pi]$



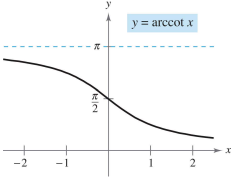
(c) Domain: $(-\infty, \infty)$, Range: $(-\pi/2, \pi/2)$



(d) Domain: $(-\infty, -1] \cup [1, \infty)$, Range: $[-\pi/2, 0) \cup (0, \pi/2]$



(e) Domain: $(-\infty, -1] \cup [1, \infty)$, Range: $[0, \pi/2) \cup (\pi/2, \pi]$



(f) Domain: $(-\infty, \infty)$, Range: $(0, \pi)$

Figure 25: Six inverse trigonometric functions.

Example 1 (Evaluating inverse trigonometric functions)

Evaluate each function.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\arccos 0$ **c.** $\arctan\sqrt{3}$ **d.** $\arcsin(0.3)$

a. By definition, $y = \arcsin\left(-\frac{1}{2}\right)$ implies that $\sin y = -\frac{1}{2}$. In the interval $[-\pi/2, \pi/2]$, the correct value of y is $-\pi/6$.

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

b. By definition, $y = \arccos 0$ implies that $\cos y = 0$. In the interval $[0, \pi]$, you have $y = \pi/2$.

$$\arccos 0 = \frac{\pi}{2}$$

c. By definition, $y = \arctan\sqrt{3}$ implies that $\tan y = \sqrt{3}$. In the interval $(-\pi/2, \pi/2)$, you have $y = \pi/3$.

$$\arctan\sqrt{3} = \frac{\pi}{3}$$

d. Using a calculator set in radian mode produces

$$\arcsin(0.3) \approx 0.305.$$

■

- Inverse functions have the properties

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

- When applying these properties to inverse trigonometric functions, remember that the trigonometric functions have inverse functions only in restricted domains.
- For x-values outside these domains, these two properties do not hold.
- For example, $\arcsin(\sin \pi)$ is equal to 0, not π.

Properties of inverse trigonometric functions If $-1 \leq x \leq 1$ and $-\pi/2 \leq y \leq \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-\infty < x < \infty$ and $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

If $|x| \geq 1$ and $0 \leq y < \pi/2$ or $\pi/2 < y \leq \pi$, then

$$\sec(\operatorname{arcsec} x) = x \quad \text{and} \quad \operatorname{arcsec}(\sec y) = y.$$

Similar properties hold for the other inverse trigonometric functions.

Example 2 (Solving an equation)

$$\arctan(2x-3)=\frac{\pi}{4} \quad \tan[\arctan(2x-3)]=\tan\frac{\pi}{4} \quad 2x-3=1 \quad x=2$$

■

Example 3 (Using right triangles)

a. Given $y=\arcsin x$, where $0<y<\pi/2$, find $\cos y$. **b.** Given $y=\operatorname{arcsec}(\sqrt{5}/2)$, find $\tan y$.

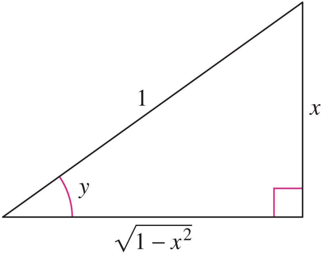
a. Because $y=\arcsin x$, you know that $\sin y=x$. This relationship between x and y can be represented by a right triangle, as shown in Figure 26.

$$\cos y=\cos(\arcsin x)=\frac{\text{adj.}}{\text{hyp.}}=\sqrt{1-x^2}$$

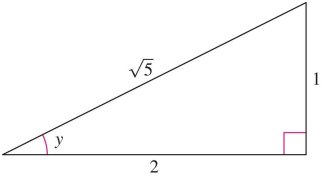
b. Use the right triangle shown in Figure 26.

$$\tan y = \tan\left[\operatorname{arcsec}\left(\frac{\sqrt{5}}{2}\right)\right] = \frac{\text{opp.}}{\text{adj.}} = \frac{1}{2}$$

■



(a) $y = \arcsin x$



(b) $y = \operatorname{arcsec}\left(\frac{\sqrt{5}}{2}\right)$

Figure 26: Using right triangles.

Derivatives of inverse trigonometric functions

- The derivative of the transcendental function $f(x) = \ln x$ is the algebraic function $f'(x) = 1/x$.
- You will now see that the derivatives of the inverse trigonometric functions also are algebraic.
- The following theorem lists the derivatives of the six inverse trigonometric functions.

Theorem 5.18 (Derivatives of inverse trigonometric functions)

Let u be a differentiable function of x.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Example 4 (Differentiating inverse trigonometric functions)

a. $\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin(2x)] = \frac{2}{\sqrt{1-(2x)^2}} = \frac{2}{\sqrt{1-4x^2}}$

b. $\frac{\mathrm{d}}{\mathrm{d}x}[\arctan(3x)] = \frac{3}{1+(3x)^2} = \frac{3}{1+9x^2}$

c. $\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin\sqrt{x}] = \frac{(1/2)x^{-1/2}}{\sqrt{1-x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}$

d. $\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arcsec} e^{2x}] = \frac{2e^{2x}}{e^{2x}\sqrt{(e^{2x})^2-1}} = \frac{2e^{2x}}{e^{2x}\sqrt{e^{4x}-1}} = \frac{2}{\sqrt{e^{4x}-1}}$

The absolute value sign is not necessary because $e^{2x} > 0$. ■

Example 5 (A derivative that can be simplified)

$$y = \arcsin x + x\sqrt{1-x^2}$$

$$y' = \frac{1}{\sqrt{1-x^2}} + x\left(\frac{1}{2}\right)(-2x)(1-x^2)^{-1/2} + \sqrt{1-x^2}$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} = \sqrt{1-x^2} + \sqrt{1-x^2} = 2\sqrt{1-x^2}$$

■

Example 6 (Analyzing an inverse trigonometric graph)

Analyze the graph of $y = (\arctan x)^2$.

$$y' = 2(\arctan x)\left(\frac{1}{1+x^2}\right) = \frac{2\arctan x}{1+x^2}$$

$$y'' = \frac{(1+x^2)\left(\frac{2}{1+x^2}\right) - (2\arctan x)(2x)}{(1+x^2)^2} = \frac{2(1-2x\arctan x)}{(1+x^2)^2}$$

- It follows that points of inflection occur when $2x\arctan x = 1$.
- Using Newton's Method, these points occur when $x \approx \pm 0.765$. Finally, because

$$\lim_{x\to\pm\infty}(\arctan x)^2 = \frac{\pi^2}{4}$$

it follows that the graph has a horizontal asymptote at $y = \pi^2/4$. The graph is shown in Figure 27. ■

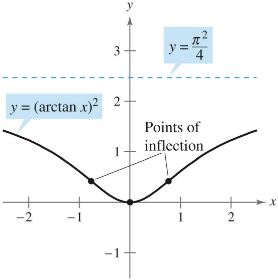


Figure 27: The graph of $y = (\arctan x)^2$ has a horizontal asymptote at $y = \pi^2/4$.

Review of basic differentiation rules

1. $\frac{\mathrm{d}}{\mathrm{d}x}[cu] = cu'$	2. $\frac{\mathrm{d}}{\mathrm{d}x}[u \pm v] = u' \pm v'$	3. $\frac{\mathrm{d}}{\mathrm{d}x}[uv] = uv' + vu'$
4. $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$	5. $\frac{\mathrm{d}}{\mathrm{d}x}[c] = 0$	6. $\frac{\mathrm{d}}{\mathrm{d}x}[u^n] = nu^{n-1}u'$
7. $\frac{\mathrm{d}}{\mathrm{d}x}[x] = 1$	8. $\frac{\mathrm{d}}{\mathrm{d}x}[\lvert u\rvert] = \frac{u}{\lvert u\rvert}(u'), \quad u \neq 0$	9. $\frac{\mathrm{d}}{\mathrm{d}x}[\ln u] = \frac{u'}{u}$
10. $\frac{\mathrm{d}}{\mathrm{d}x}[e^u] = e^u u'$	11. $\frac{\mathrm{d}}{\mathrm{d}x}[\log_a u] = \frac{u'}{(\ln a)u}$	12. $\frac{\mathrm{d}}{\mathrm{d}x}[a^u] = (\ln a)a^u u'$
13. $\frac{\mathrm{d}}{\mathrm{d}x}[\sin u] = (\cos u)u'$	14. $\frac{\mathrm{d}}{\mathrm{d}x}[\cos u] = -(\sin u)u'$	15. $\frac{\mathrm{d}}{\mathrm{d}x}[\tan u] = (\sec^2 u)u'$
16. $\frac{\mathrm{d}}{\mathrm{d}x}[\cot u] = -(\csc^2 u)u'$	17. $\frac{\mathrm{d}}{\mathrm{d}x}[\sec u] = (\sec u \tan u)u'$	18. $\frac{\mathrm{d}}{\mathrm{d}x}[\csc u] = -(\csc u \cot u)u'$
19. $\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$	20. $\frac{\mathrm{d}}{\mathrm{d}x}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$	21. $\frac{\mathrm{d}}{\mathrm{d}x}[\arctan u] = \frac{u'}{1+u^2}$
22. $\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$	23. $\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arcsec} u] = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$	24. $\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arccsc} u] = \frac{-u'}{\lvert u\rvert\sqrt{u^2-1}}$

Table of Contents

Integrals involving inverse trigonometric functions

- The derivatives of the six inverse trigonometric functions fall into three pairs. In each pair, the derivative of one function is the negative of the other.
- For example

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}.$$

- When listing the antiderivative that corresponds to each of the inverse trigonometric functions, you need to use only one member from each pair.
- It is conventional to use $\arcsin x$ as the antiderivative of $1/\sqrt{1-x^2}$, rather than $-\arccos x$.

Identities involving inverse trigonometric functions

$$\arcsin x + \arccos x = \frac{1}{2}\pi, \quad |x| \le 1$$
$$\arctan x + \operatorname{arccot} x = \frac{1}{2}\pi, \quad |x| \in \mathbb{R}$$
$$\operatorname{arcsec} x + \operatorname{arccsc} x = \frac{1}{2}\pi, \quad |x| \ge 1$$

Theorem 5.19 (Integrals involving inverse trigonometric functions)

Let u be a differentiable function of x, and let $a > 0$.
1. $\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C$ **2.** $\int \frac{\mathrm{d}u}{a^2+u^2} = \frac{1}{a}\arctan \frac{u}{a} + C$ **3.** $\int \frac{\mathrm{d}u}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec} \frac{|u|}{a} + C$

Example 1 (Integration with inverse trigonometric functions)

a. $\int \frac{\mathrm{d}x}{\sqrt{4-x^2}} = \arcsin \frac{x}{2} + C, \quad u = x,\ a = 2$

b. $\int \frac{\mathrm{d}x}{2+9x^2} = \frac{1}{3} \int \frac{3\,\mathrm{d}x}{(\sqrt{2})^2+(3x)^2} = \frac{1}{3\sqrt{2}} \arctan \frac{3x}{\sqrt{2}} + C, \quad u = 3x,\ a = \sqrt{2}$

c. $\int \frac{\mathrm{d}x}{x\sqrt{4x^2-9}} = \int \frac{2\,\mathrm{d}x}{(2x)\sqrt{(2x)^2-3^2}} = \frac{1}{3} \operatorname{arcsec} \frac{|2x|}{3} + C, \quad u = 2x,\ a = 3$ ■

Example 2 (Integration by substitution)

Find $\int \frac{\mathrm{d}x}{\sqrt{e^{2x}-1}}$.

$$\int \frac{\mathrm{d}x}{\sqrt{e^{2x}-1}} = \int \frac{\mathrm{d}x}{\sqrt{(e^x)^2-1}} = \int \frac{\mathrm{d}u/u}{\sqrt{u^2-1}}$$
$$= \int \frac{\mathrm{d}u}{u\sqrt{u^2-1}} = \operatorname{arcsec} \frac{|u|}{1} + C = \operatorname{arcsec} e^x + C \qquad \blacksquare$$

Example 3 (Rewriting as the sum of two quotients)

Find $\int \frac{x+2}{\sqrt{4-x^2}}\,\mathrm{d}x$.

$$\begin{aligned}
\int \frac{x+2}{\sqrt{4-x^2}}\,\mathrm{d}x &= \int \frac{x}{\sqrt{4-x^2}}\,\mathrm{d}x + \int \frac{2}{\sqrt{4-x^2}}\,\mathrm{d}x \\
&= -\frac{1}{2}\int \frac{-2x}{\sqrt{4-x^2}}\,\mathrm{d}x + 2\int \frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x \\
&= -\frac{1}{2}\left[\frac{(4-x^2)^{1/2}}{1/2}\right] + 2\arcsin\frac{x}{2} + C \\
&= -\sqrt{4-x^2} + 2\arcsin\frac{x}{2} + C \qquad \blacksquare
\end{aligned}$$

Completing the square

- Completing the square helps when quadratic functions are involved in the integrand.
- For example, the quadratic $x^2 + bx + c$ can be written as the difference of two squares by adding and subtracting $(b/2)^2$.

$$\begin{aligned} x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \end{aligned}$$

Example 4 (Completing the square)

Find $\int \frac{\mathrm{d}x}{x^2-4x+7}$.

- You can write the denominator as the sum of two squares, as follows.

$$x^2 - 4x + 7 = (x^2 - 4x + 4) - 4 + 7 = (x-2)^2 + 3 = u^2 + a^2$$

- Now, in this completed square form, let $u = x - 2$ and $a = \sqrt{3}$.

$$\int \frac{\mathrm{d}x}{x^2 - 4x + 7} = \int \frac{\mathrm{d}x}{(x-2)^2 + 3} = \frac{1}{\sqrt{3}} \arctan \frac{x-2}{\sqrt{3}} + C \qquad \blacksquare$$

Example 5 (Completing the square (negative leading coefficient))

Find the area of the region bounded by the graph of $f(x) = \frac{1}{\sqrt{3x - x^2}}$ the x-axis, and the lines $x = \frac{3}{2}$ and $x = \frac{9}{4}$.

In Figure 28, you can see that the area is given by

$$\text{Area} = \int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}} \,\mathrm{d}x = \int_{3/2}^{9/4} \frac{\mathrm{d}x}{\sqrt{(3/2)^2 - [x - (3/2)]^2}}$$

$$= \arcsin \left[\frac{x - (3/2)}{3/2} \right]_{3/2}^{9/4} = \arcsin \frac{1}{2} - \arcsin 0 = \frac{\pi}{6} \approx 0.524 \qquad \blacksquare$$

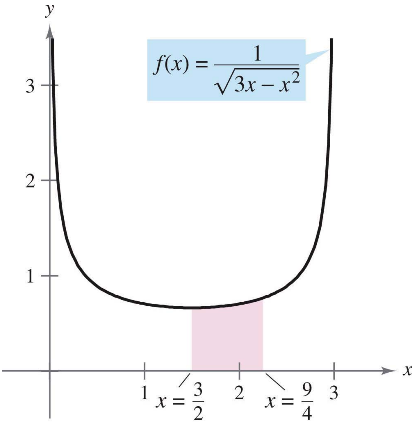


Figure 28: The area of the region bounded by the graph of f, the x-axis, and the lines $x = \frac{3}{2}$ and $x = \frac{9}{4}$ is $\pi/6$.

Review of basic integration rules

Table 2: Basic integration rules ($a > 0$)

1. $\int kf(u)\,\mathrm{d}u = k\int f(u)\,\mathrm{d}u$	2. $\int [f(u) \pm g(u)]\,\mathrm{d}u = \int f(u)\,\mathrm{d}u \pm \int g(u)\,\mathrm{d}u$
3. $\int \mathrm{d}u = u + C$	4. $\int u^n\,\mathrm{d}u = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
5. $\int \frac{\mathrm{d}u}{u} = \ln\lvert u\rvert + C$	6. $\int e^u\,\mathrm{d}u = e^u + C$
7. $\int a^u\,\mathrm{d}u = \left(\frac{1}{\ln a}\right) a^u + C$	8. $\int \sin u\,\mathrm{d}u = -\cos u + C$
9. $\int \cos u\,\mathrm{d}u = \sin u + C$	10. $\int \tan u\,\mathrm{d}u = -\ln\lvert\cos u\rvert + C$
11. $\int \cot u\,\mathrm{d}u = \ln\lvert\sin u\rvert + C$	12. $\int \sec u\,\mathrm{d}u = \ln\lvert\sec u + \tan u\rvert + C$
13. $\int \csc u\,\mathrm{d}u = -\ln\lvert\csc u + \cot u\rvert + C$	14. $\int \sec^2 u\,\mathrm{d}u = \tan u + C$
15. $\int \csc^2 u\,\mathrm{d}u = -\cot u + C$	16. $\int \sec u \tan u\,\mathrm{d}u = \sec u + C$
17. $\int \csc u \cot u\,\mathrm{d}u = -\csc u + C$	18. $\int \frac{\mathrm{d}u}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$
19. $\int \frac{\mathrm{d}u}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$	20. $\int \frac{\mathrm{d}u}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{\lvert u\rvert}{a} + C$

Example 6 (Comparing integration problems)

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}}$ **b.** $\int \frac{x\,\mathrm{d}x}{\sqrt{x^2-1}}$ **c.** $\int \frac{\mathrm{d}x}{\sqrt{x^2-1}}$

a. You can find this integral (it fits the Arcsecant Rule).

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}} = \operatorname{arcsec}|x| + C$$

b. You can find this integral (it fits the Power Rule).

$$\begin{aligned}\int \frac{x\,\mathrm{d}x}{\sqrt{x^2-1}} &= \frac{1}{2}\int (x^2-1)^{-1/2}(2x)\,\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{(x^2-1)^{1/2}}{1/2}\right] + C = \sqrt{x^2-1} + C\end{aligned}$$

c. You cannot find this integral using the techniques you have studied so far.

Example 7 (Comparing integration problems)

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\int \frac{\mathrm{d}x}{x \ln x}$ **b.** $\int \frac{\ln x \,\mathrm{d}x}{x}$ **c.** $\int \ln x \,\mathrm{d}x$

a. You can find this integral (it fits the Log Rule).

$$\int \frac{\mathrm{d}x}{x \ln x} = \int \frac{1/x}{\ln x}\,\mathrm{d}x = \ln|\ln x| + C$$

b. You can find this integral (it fits the Power Rule).

$$\int \frac{\ln x \,\mathrm{d}x}{x} = \int \left(\frac{1}{x}\right)(\ln x)\,\mathrm{d}x = \frac{(\ln x)^2}{2} + C$$

c. You cannot find this integral using the techniques you have studied so far. ■