Chapter 4 Integration

Szu-Chi Chung

Department of Applied Mathematics, National Sun Yat-sen University

October 22, 2021

Table of Contents

Table of Contents

Antiderivatives

- To find a function F whose derivative is $f(x) = 3x^2$, you might use your knowledge of derivatives to conclude that

$$F(x) = x^3 \quad \text{because} \quad \frac{\mathrm{d}}{\mathrm{d}x}[x^3] = 3x^2.$$

The function F is an antiderivative of f.

Definition 4.1 (Antiderivative)

A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem 4.1 (Representation of antiderivatives)

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$, for all x in I where C is a constant.

- The proof of Theorem 4.1 in one direction is straightforward. That is, if $G(x) = F(x) + C$, $F'(x) = f(x)$, and C is a constant, then

$$G'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[F(x) + C] = F'(x) + 0 = f(x).$$

- To prove this theorem in the other direction, assume that G is an antiderivative of f.
- Define a function H such that $H(x) = G(x) - F(x)$. For any two points a and b ($a < b$) in the interval, H is continuous on $[a, b]$ and differentiable on (a, b).
- By the Mean Value Theorem,

$$H'(c) = \frac{H(b) - H(a)}{b - a}$$

for some c in (a, b). However, $H'(c) = 0$, so $H(a) = H(b)$.
- Because a and b are arbitrary points in the interval, you know that H is a constant function C. So, $G(x) - F(x) = C$ and it follows that $G(x) = F(x) + C$. □

- You can represent the entire family of antiderivatives of a function by adding a constant to a known antiderivative.
- For example, knowing that $D_x[x^2] = 2x$, you can represent the family of all antiderivatives of $f(x) = 2x$ by

$$G(x) = x^2 + C \qquad \text{Family of all antiderivatives of } f(x) = 2x$$

  where C is a constant. The constant C is called the constant of integration.
- The family of functions represented by G is the general antiderivative of f, and $G(x) = x^2 + C$ is the general solution of the differential equation

$$G'(x) = 2x. \qquad \text{Differential equation}$$

- A differential equation in x and y is an equation that involves x, y, and derivatives of y. For instance, $y' = 3x$ and $y' = x^2 + 1$ are examples of differential equations.

Example 1 (Solving a differential equation)

Find the general solution of the differential equation $\frac{dy}{dx} = 2$.

- To begin, you need to find a function whose derivative is 2.
- One such function is

$$y = 2x. \qquad 2x \text{ is an antiderivative of } 2$$

- Now, you can use Theorem 4.1 to conclude that the general solution of the differential equation is

$$y = 2x + C. \qquad \text{General solution}$$

- The graphs of several functions of the form $y = 2x + C$ are shown in Figure 1. ■

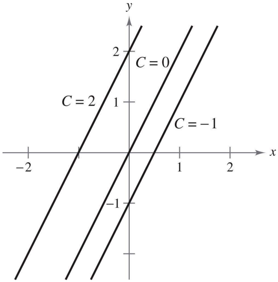


Figure 1: $y' = 2$: $y = 2x + C$, $C = -1,\ 0,\ 2.$

- When solving a differential equation of the form

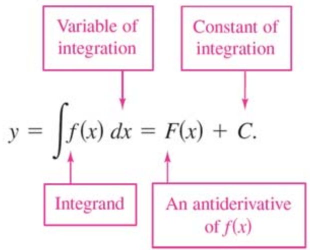
$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

it is convenient to write it in the equivalent differential form

$$\mathrm{d}y = f(x)\,\mathrm{d}x.$$

- The operation of finding all solutions of this equation is called antidifferentiation (or indefinite integration) and is denoted by an integral sign.

- The general solution is denoted by antiderivative.



- The expression $\int f(x)\,\mathrm{d}x$ is read as the antiderivative of f with respect to x. So, the differential $\mathrm{d}x$ serves to identify x as the variable of integration. The term indefinite integral is a synonym for antiderivative.

Basic integration rules

- The inverse nature of integration and differentiation can be verified by substituting $F'(x)$ for $f(x)$ in the indefinite integration definition to obtain
$$\int F'(x)\,\mathrm{d}x = F(x) + C.$$
Integration is the "inverse" of differentiation
- Moreover, if $\int f(x)\,\mathrm{d}x = F(x) + C$, then
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int f(x)\,\mathrm{d}x\right] = f(x).$$
Differentiation is the "inverse" of integration
- These two equations allow you to obtain integration formulas directly from differentiation formulas, as shown in the following summary.
- For instance, check out `https://www.mathdoubts.com/integral-sum-rule-proof/` for the sum rule

Differentiation Formula	Integration Formula
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$ Power Rule
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$

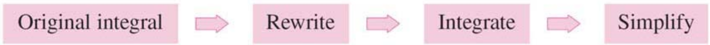
Example 2 (Applying the basic integration rules)

Describe the antiderivatives of $3x$.

$$\int 3x\,\mathrm{d}x = 3\int x\,\mathrm{d}x = 3\int x^1\,\mathrm{d}x = 3\left(\frac{x^2}{2}\right) + C = \frac{3}{2}x^2 + C$$

So, the antiderivatives of $3x$ are of the form $\frac{3}{2}x^2 + C$ where C is any constant. ∎

- Note that the general pattern of integration is similar to that of differentiation.



Example 3 (Rewriting before integrating)

	Original Integral	Rewrite	Integrate	Simplify
a.	$\int \frac{1}{x^3}\,dx$	$\int x^{-3}\,dx$	$\frac{x^{-2}}{-2} + C$	$-\frac{1}{2x^2} + C$
b.	$\int \sqrt{x}\,dx$	$\int x^{1/2}\,dx$	$\frac{x^{3/2}}{3/2} + C$	$\frac{2}{3}x^{3/2} + C$
c.	$\int 2\sin x\,dx$	$2\int \sin x\,dx$	$2(-\cos x) + C$	$-2\cos x + C$

■

Example 4 (Integrating polynomial functions)

a. $$\int dx = \int 1\,dx = x + C$$

b. $$\int (x+2)\,dx = \int x\,dx + \int 2\,dx$$

$$= \frac{x^2}{2} + C_1 + 2x + C_2 = \frac{x^2}{2} + 2x + C$$

c. $$\int (3x^4 - 5x^2 + x)\,dx = 3\left(\frac{x^5}{5}\right) - 5\left(\frac{x^3}{3}\right) + \frac{x^2}{2} + C$$

$$= \frac{3}{5}x^5 - \frac{5}{3}x^3 + \frac{1}{2}x^2 + C$$

■

Example 5 (Rewriting before integrating)

$$\int \frac{x+1}{\sqrt{x}}\,dx = \int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx$$

$$= \int \left(x^{1/2} + x^{-1/2}\right) dx = \frac{2}{3}x^{3/2} + 2x^{1/2} + C \quad \blacksquare$$

Example 6 (Rewriting before integrating)

$$\int \frac{\sin x}{\cos^2 x}\,dx = \int \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) dx = \int \sec x \tan x\,dx = \sec x + C \blacksquare$$

Example 7 (Rewriting before integrating)

	Original Integral	Rewrite	Integrate	Simplify
a.	$\int \frac{2}{\sqrt{x}}\,dx$	$2\int x^{-1/2}\,dx$	$2\left(\frac{x^{1/2}}{1/2}\right) + C$	$4x^{1/2} + C$
b.	$\int (t^2+1)^2\,dt$	$\int (t^4 + 2t^2 + 1)\,dt$	$\frac{t^5}{5} + 2\left(\frac{t^3}{3}\right) + t + C$	$\frac{1}{5}t^5 + \frac{2}{3}t^3 + t + C$
c.	$\int \frac{x^3+3}{x^2}\,dx$	$\int (x + 3x^{-2})\,dx$	$\frac{x^2}{2} + 3\left(\frac{x^{-1}}{-1}\right) + C$	$\frac{1}{2}x^2 - \frac{3}{x} + C$
d.	$\int \sqrt[3]{x}(x-4)\,dx$	$\int (x^{4/3} - 4x^{1/3})\,dx$	$\frac{x^{7/3}}{7/3} - 4\left(\frac{x^{4/3}}{4/3}\right) + C$	$\frac{3}{7}x^{7/3} - 3x^{4/3} + C$ $\blacksquare$

Initial conditions and particular solutions

- You have already seen that the equation $y = \int f(x)\,\mathrm{d}x$ has many solutions (each differing from the others by a constant).
- This means that the graphs of any two antiderivatives of f are vertical translations of each other. For example, Figure 2 shows the graphs of several antiderivatives of the form

$$y = \int (3x^2 - 1)\,\mathrm{d}x = x^3 - x + C \quad \text{General solution}$$

  for various integer values of C.
- Each of these antiderivatives is a solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 1.$$

- In many applications of integration, you are given enough information to determine a particular solution. To do this, you need only know the value of $y = F(x)$ for one value of x. This information is called an initial condition.

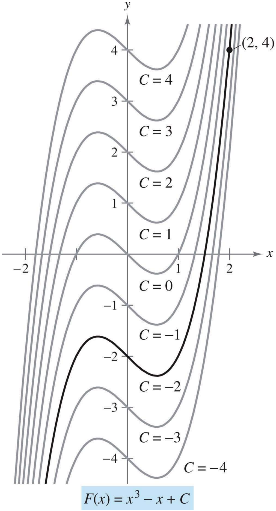


Figure 2: $y = \int (3x^2 - 1)\,\mathrm{d}x = x^3 - x + C, \;\; C = -1, \; 0, \; 1, \; 2, \; 3, \; 4.$

- For example, in Figure 2, only one curve passes through the point $(2, 4)$.

- To find this curve, you can use the following information.

$$F(x) = x^3 - x + C \qquad\qquad F(2) = 4.$$

- By using the initial condition in the general solution, you can determine that $F(2) = 8 - 2 + C = 4$, which implies that $C = -2$. So, you obtain

$$F(x) = x^3 - x - 2. \qquad \text{Particular solution}$$

Example 8 (Finding a particular solution)

Find the general solution of $F'(x) = \frac{1}{x^2}$, $x > 0$ and find the particular solution that satisfies the initial condition $F(1) = 0$.

- To find the general solution, integrate to obtain

$$F(x) = \int \frac{1}{x^2}\,\mathrm{d}x = \int x^{-2}\,\mathrm{d}x = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C, \quad x > 0.$$

- Using the initial condition $F(1) = 0$, you can solve for C as follows.

$$F(1) = -\frac{1}{1} + C = 0 \quad \Longrightarrow \quad C = 1$$

- So, the particular solution, as shown in Figure 3, is

$$F(x) = -\frac{1}{x} + 1, \quad x > 0. \qquad \text{Particular solution}$$

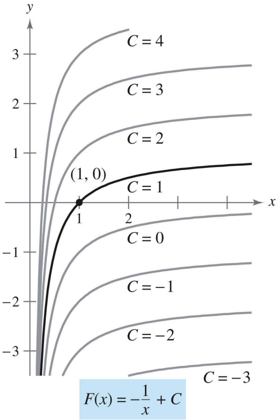


Figure 3: The particular solution of $F'(x) = \frac{1}{x^2}$ that satisfies the initial condition $F(1) = 0$ is $F(x) = -(1/x) + 1$, $x > 0$.

Table of Contents

Sigma notation

- This section begins by introducing a concise notation for sums. This notation is called sigma notation because it uses the uppercase Greek letter sigma, written as $\sum$.

Definition 4.2 (Sigma notation)

The sum of n terms $a_1, a_2, a_3, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the index of summation, a_i is the ith term of the sum, and the upper and lower bounds of summation are n and 1.

Example 1 (Examples of sigma notation)

a. $\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6$

b. $\sum_{i=0}^{5}(i+1) = 1 + 2 + 3 + 4 + 5 + 6$

c. $\sum_{j=3}^{7} j^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2$

d. $\sum_{j=1}^{5} \frac{1}{\sqrt{j}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}}$

e. $\sum_{k=1}^{n} \frac{1}{n}(k^2+1) = \frac{1}{n}(1^2+1) + \frac{1}{n}(2^2+1) + \cdots + \frac{1}{n}(n^2+1)$

f. $\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$ ■

The following properties of summation can be derived using the associative and commutative properties of addition and the distributive property of addition over multiplication. (In the first property, k is a constant.)

1. $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$
2. $\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Theorem 4.2 (Summation formulas)

1. $\sum_{i=1}^{n} c = cn, c$ *is constant*
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2 = \frac{n^2(n+1)^2}{4}$

Example 2 (Evaluating a sum)

Evaluate $\sum_{i=1}^{n} \frac{i+1}{n^2}$ for $n = 10$, 100, 1000 and 10000.

- Applying Theorem 4.2, you can write

$$\sum_{i=1}^{n} \frac{i+1}{n^2} = \frac{1}{n^2} \sum_{i=1}^{n} (i+1) = \frac{1}{n^2} \left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \right)$$
$$= \frac{1}{n^2} \left[\frac{n(n+1)}{2} + n \right] = \frac{1}{n^2} \left[\frac{n^2+3n}{2} \right] = \frac{n+3}{2n}.$$

- Now you can evaluate the sum by substituting the appropriate values of n, as shown in the table. ■

n	$\sum_{i=1}^{n} \frac{i+1}{n^2} = \frac{n+3}{2n}$
10	0.65000
100	0.51500
1000	0.50150
10000	0.50015

- In Euclidean geometry, the simplest type of plane region is a rectangle. Although people often say that the formula for the area of a rectangle is $A = bh$, it is actually more proper to say that this is the definition of the area of a rectangle.
- From this definition, you can develop formulas for the areas of many other plane regions. For example, to determine the area of a triangle, you can form a rectangle whose area is twice that of the triangle, as shown in Figure 4.

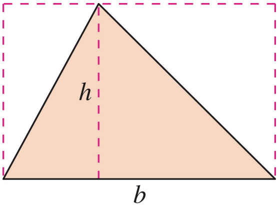
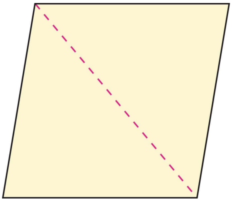
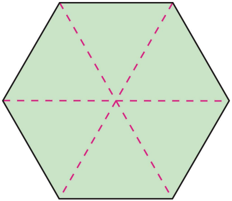


Figure 4: Area of triangle: $A = \frac{1}{2}bh$

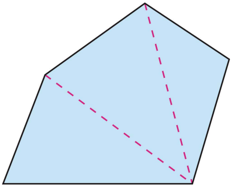
- Once you know how to find the area of a triangle, you can determine the area of any polygon by subdividing the polygon into triangular regions, as shown in Figure 5.



(a) Parallelogram



(b) Hexagon



(c) Polygon

Figure 5: Determine the area of any polygon by subdividing the polygon into triangular regions.

- Finding the areas of regions other than polygons is more difficult. The ancient Greeks were able to determine formulas for the areas of some general regions (principally those bounded by conics) by the exhaustion method.
- The clearest description of this method was given by Archimedes (287-212 B.C.) (the greatest applied mathematician of antiquity). Essentially, the method is a limiting process in which the area is squeezed between two polygons—one inscribed in the region and one circumscribed about the region.
- For instance, in Figure 6 the area of a circular region is approximated by an n-sided inscribed polygon and an n-sided circumscribed polygon.

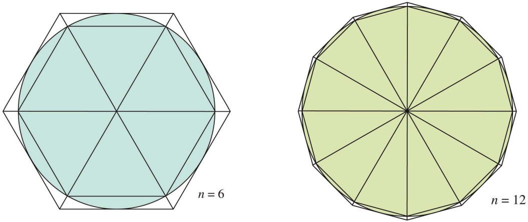
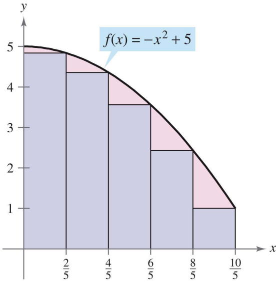


Figure 6: The exhaustion method for finding the area of a circular region.

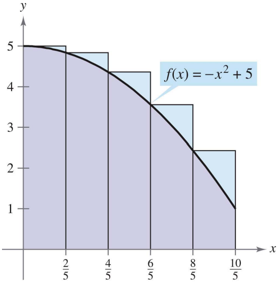
- For each value of n, the area of the inscribed polygon is less than the area of the circle, and the area of the circumscribed polygon is greater than the area of the circle.
- Moreover, as n increases, the areas of both polygons become better and better approximations of the area of the circle.

Example 3 (Approximating the area of a plane region)

Use the five rectangles in Figure 7 to find two approximations of the area of the region lying between the graph of $f(x) = -x^2 + 5$ and the x-axis between $x = 0$ and $x = 2$.



(a) The area of the parabolic region is greater than the area of the rectangles



(b) The area of the parabolic region is less than the area of the rectangles.

Figure 7: The Exhaustion Method for finding the area of a parabolic region.

a. The right endpoints of the five intervals are $\frac{2}{5}i$, where $i = 1, 2, 3, 4, 5$.

- The width of each rectangle is $\frac{2}{5}$, and the height of each rectangle can be obtained by evaluating f at the right endpoint of each interval.

$$\left[0, \frac{2}{5}\right], \left[\frac{2}{5}, \frac{4}{5}\right], \left[\frac{4}{5}, \frac{6}{5}\right], \left[\frac{6}{5}, \frac{8}{5}\right], \left[\frac{8}{5}, \frac{10}{5}\right]$$

Evaluate f at the right endpoints of these intervals.

- The sum of the areas of the five rectangles is

$$\sum_{i=1}^{5} \overbrace{f\left(\frac{2i}{5}\right)}^{\text{Height}} \overbrace{\left(\frac{2}{5}\right)}^{\text{Width}} = \sum_{i=1}^{5} \left[-\left(\frac{2i}{5}\right)^2 + 5\right] \left(\frac{2}{5}\right) = \frac{162}{25} = 6.48.$$

- Because each of the five rectangles lies inside the parabolic region, you can conclude that the area of the parabolic region is greater than 6.48.

b. The left endpoints of the five intervals are $\frac{2}{5}(i-1)$, where $i = 1$, 2, 3, 4, 5. The width of each rectangle is $\frac{2}{5}$, and the height of each rectangle can be obtained by evaluating f at the left endpoint of each interval.

- So, the sum is

$$\sum_{i=1}^{5} \overbrace{f\left(\frac{2i-2}{5}\right)}^{\text{Height}} \overbrace{\left(\frac{2}{5}\right)}^{\text{Width}} = \sum_{i=1}^{5} \left[-\left(\frac{2i-2}{5}\right)^2 + 5\right] \left(\frac{2}{5}\right) = \frac{202}{25} = 8.08.$$

- Because the parabolic region lies within the union of the five rectangular regions, you can conclude that the area of the parabolic region is less than 8.08.
- By combining the results in parts (a) and (b), you can conclude that $6.48 < (\text{Area of region}) < 8.08$. ■

- Consider a plane region bounded above by the graph of a nonnegative, continuous function $y = f(x)$, as shown in Figure 8.
- The region is bounded below by the x-axis, and the left and right boundaries of the region are the vertical lines $x = a$ and $x = b$.

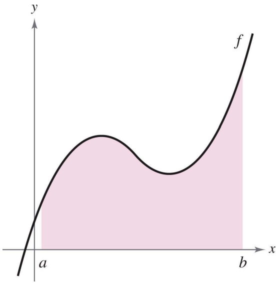


Figure 8: The region under a curve.

- To approximate the area of the region, begin by subdividing the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$, as shown in Figure 9.

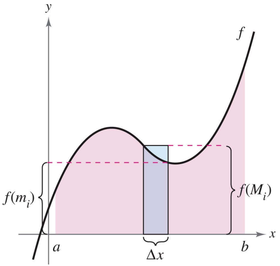


Figure 9: The interval $[a, b]$ is divided into n subintervals of width $\Delta x = \frac{b-a}{n}$.

- The endpoints of the intervals are as follows.

$$\overbrace{a + 0(\Delta x)}^{a = x_0} < \overbrace{a + 1(\Delta x)}^{x_1} < \overbrace{a + 2(\Delta x)}^{x_2} < \cdots < \overbrace{a + n(\Delta x)}^{x_n = b}$$

- Because f is continuous, the Extreme Value Theorem guarantees the existence of a minimum and a maximum value of $f(x)$ in each subinterval.

$$f(m_i) = \text{Minimum value of } f(x) \text{ in } i\text{th subinterval}$$
$$f(M_i) = \text{Maximum value of } f(x) \text{ in } i\text{th subinterval}$$

- Next, define an inscribed rectangle lying inside the ith subregion and a circumscribed rectangle extending outside the ith subregion. The height of the ith inscribed rectangle is $f(m_i)$ and the height of the ith circumscribed rectangle is $f(M_i)$.
- For each i, the area of the inscribed rectangle is less than or equal to the area of the circumscribed rectangle.

$$\begin{pmatrix} \text{Area of inscribed} \\ \text{rectangle} \end{pmatrix} = f(m_i)\Delta x \leq f(M_i)\Delta x = \begin{pmatrix} \text{Area of circumscribed} \\ \text{rectangle} \end{pmatrix}$$

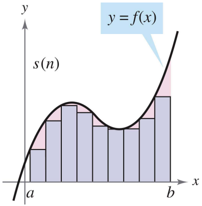
- The sum of the areas of the inscribed rectangles is called a lower sum, and the sum of the areas of the circumscribed rectangles is called an upper sum.

$$\text{Lower sum} = s(n) = \sum_{i=1}^{n} f(m_i)\Delta x \qquad \text{Area of inscribed rectangles}$$

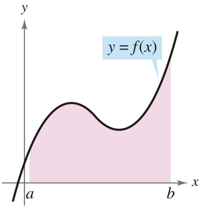
$$\text{Upper sum} = S(n) = \sum_{i=1}^{n} f(M_i)\Delta x \qquad \text{Area of circumscribed rectangles}$$

- You can see that the lower sum $s(n)$ is less than or equal to the upper sum $S(n)$. Moreover, the actual area of the region lies between these two sums.

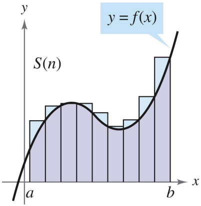
$$s(n) \le (\text{Area of region}) \le S(n)$$



(a) Area of inscribed rectangles is less than area of region.



(b) Area of region.

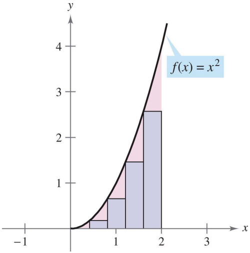


(c) Area of circumscribed rectangles is greater than area of region.

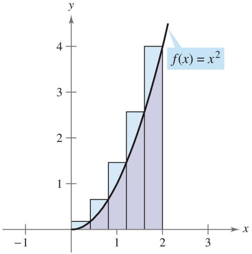
Example 4 (Finding upper and lower sums for a region)

Find the upper and lower sums for the region bounded by the graph of $f(x) = x^2$ and the x-axis between $x = 0$ and $x = 2$.

- To begin, partition the interval $[0, 2]$ into n subintervals, each of width
$$\Delta x = \frac{b - a}{n} = \frac{2 - 0}{n} = \frac{2}{n}.$$
- The following figure shows the endpoints of the subintervals and several inscribed and circumscribed rectangles.



(a) Inscribed rectangles.



(b) Circumscribed rectangles

- Because f is increasing on the interval $[0, 2]$, the minimum value on each subinterval occurs at the left endpoint, and the maximum value occurs at the right endpoint.

Left Endpoints	Right Endpoints
$m_i = 0 + (i-1)\left(\frac{2}{n}\right) = \frac{2(i-1)}{n}$	$M_i = 0 + (i)\left(\frac{2}{n}\right) = \frac{2i}{n}$

- Using the left endpoints, the lower sum is

$$\begin{aligned} s(n) &= \sum_{i=1}^{n} f(m_i)\Delta x = \sum_{i=1}^{n} f\left[\frac{2(i-1)}{n}\right]\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n}\left[\frac{2(i-1)}{n}\right]^2\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left(\frac{8}{n^3}\right)(i^2 - 2i + 1) \\ &= \frac{8}{n^3}\left(\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right) \\ &= \frac{8}{n^3}\left\{\frac{n(n+1)(2n+1)}{6} - 2\left[\frac{n(n+1)}{2}\right] + n\right\} \\ &= \frac{4}{3n^3}(2n^3 - 3n^2 + n) = \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}. \end{aligned}$$

- Using the right endpoints, the upper sum is

$$\begin{aligned} S(n) &= \sum_{i=1}^{n} f(M_i)\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \left(\frac{2}{n}\right) = \sum_{i=1}^{n} \left(\frac{8}{n^3}\right) i^2 \\ &= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\ &= \frac{4}{3n^3}(2n^3+3n^2+n) = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}. \end{aligned}$$

■

Theorem 4.3 (Limits of the lower and upper sums)

Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\Delta x = \lim_{n\to\infty} S(n)$$

where $\Delta x = (b-a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

- You are free to choose an arbitrary x-value in the ith subinterval, as in the following definition of the area of a region in the plane.

Definition 4.3 (The area of a region in the plane)

Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \le c_i \le x_i$$

where $\Delta x = (b - a)/n$ (see Figure 12).

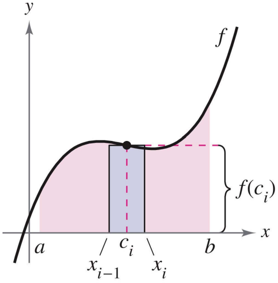


Figure 12: Area of the limit definition.

Example 5 (Finding area by the limit definition)

Find the area of the region bounded by the graph $f(x) = x^3$, the x-axis, and the vertical lines $x = 0$ and $x = 1$ as shown in Figure 13.

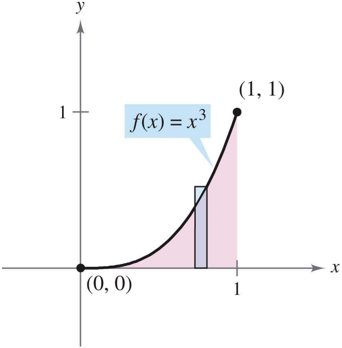


Figure 13: $f(x) = x^3,\ 0 \leq x \leq 1.$

- Begin by noting that f is continuous and nonnegative on the interval $[0, 1]$. Next, partition the interval $[0, 1]$ into n subintervals, each of width $\Delta x = 1/n$.

- According to the definition of area, you can choose any x-value in the ith subinterval.
- For this example, the right endpoints $c_i = i/n$ are convenient.

$$\begin{aligned}\text{Area} &= \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n} i^3 \\ &= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] = \lim_{n\to\infty}\left(\frac{1}{4}+\frac{1}{2n}+\frac{1}{4n^2}\right) = \frac{1}{4}\end{aligned}$$

- The area of the region is $\frac{1}{4}$. ■

Example 6 (Finding area by the limit definition)

Find the area of the region bounded by the graph of $f(x) = 4 - x^2$, the x-axis, and the vertical lines $x = 1$ and $x = 2$, as shown in Figure 14.

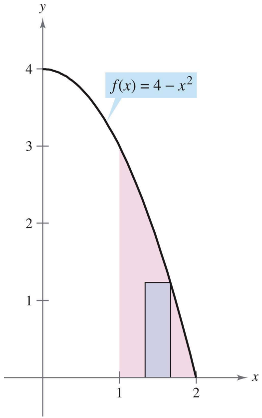


Figure 14: The area of the region bounded by the graph of f, the x-axis, $x = 1$, and $x = 2$ is $\frac{5}{3}$.

The function f is continuous and nonnegative on the interval $[1,2]$ and so begin by partitioning the interval into n subintervals, each of width $\Delta x = 1/n$. Choosing the right endpoint

$$c_i = a + i\Delta x = 1 + \frac{i}{n}$$

of each subinterval, you obtain

$$\begin{aligned}
\text{Area} &= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\to\infty} \sum_{i=1}^{n} \left(3 - \frac{2i}{n} - \frac{i^2}{n^2}\right)\left(\frac{1}{n}\right) \\
&= \lim_{n\to\infty} \left(\frac{1}{n}\sum_{i=1}^{n} 3 - \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^3}\sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to\infty} \left[3 - \left(1 + \frac{1}{n}\right) - \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)\right] = 3 - 1 - \frac{1}{3} = \frac{5}{3}.
\end{aligned}$$

The area of the region is $\frac{5}{3}$. ■

Table of Contents

Riemann sums

- In previous section, we partition the axis using equal width. In fact we can use a partition having subintervals of unequal widths as shown in Figure 15.
- The reason this strategy also gave the proper area is that as n increases, the width of the largest subinterval approaches zero.
- This is a key feature of the development of definite integrals.

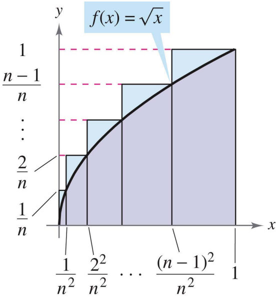


Figure 15: A partition with subintervals of unequal widths.

Definition 4.4 (Riemann sum)

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval. If c_i is any point in the ith subinterval $[x_{i-1}, x_i]$, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a Riemann sum of f for the partition Δ.

- The width of the largest subinterval of a partition Δ is the norm of the partition and is denoted by $\|\Delta\|$. If every subinterval is of equal width, the partition is regular and the norm is denoted by

$$\|\Delta\| = \Delta x = \frac{b-a}{n}. \qquad \text{regular partition}$$

- For a general partition, the norm is related to the number of subintervals of $[a, b]$ in the following way.

$$\frac{b-a}{\|\Delta\|} \leq n \qquad \text{general partition}$$

- So, the number of subintervals in a partition approaches infinity as the norm of the partition approaches 0. That is, $\|\Delta\| \to 0$ implies that $n \to \infty$.
- The converse of this statement is not true. For example, let Δ_n be the partition of the interval $[0, 1]$ given by

$$0 < \frac{1}{2^n} < \frac{1}{2^{n-1}} < \cdots < \frac{1}{8} < \frac{1}{4} < \frac{1}{2} < 1.$$

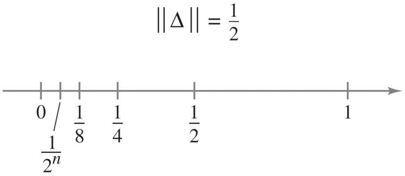


Figure 16: $n \to \infty$ does not imply that $\|\Delta\| \to 0$.

- As shown in Figure 16, for any positive value of n, the norm of the partition Δ_n is $\frac{1}{2}$.
- So, letting n approach infinity does not force $\|\Delta\|$ to approach 0. In a regular partition, however, the statements $\|\Delta\| \to 0$ and $n \to \infty$ are equivalent.
- Now we are ready to define the definite integral, consider the following limit.

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i = L$$

- To say that this limit exists means there exists a real number L such that for each $\epsilon > 0$ there exists a $\delta > 0$ so that for every partition with $\|\Delta\| < \delta$ it follows that

$$\left| L - \sum_{i=1}^{n} f(c_i) \Delta x_i \right| < \epsilon$$

regardless of the choice of c_i in the ith subinterval of each partition Δ.

Definite integrals

Definition 4.5 (Definite integral)

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

exists (as described above), then f is said to be integrable on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,\mathrm{d}x.$$

The limit is called the definite integral of f from a to b. The number a is the lower limit of integration, and the number b is the upper limit of integration.

Four steps of finding the definite integral $\int_a^b f(x)\,\mathrm{d}x$ using Riemann sum

1. Partition: $a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$
2. Sampling: $c_i \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$
3. Summation: $\sum_{i=1}^{n} f(c_i)\Delta x_i$
4. Limit: $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,\mathrm{d}x$

Theorem 4.4 (Continuity implies integrability)

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\,\mathrm{d}x$ exists.

Example 2 (Evaluating a definite integral as a limit)

Evaluate the definite integral $\int_{-2}^{1} 2x \, \mathrm{d}x$.

- The function $f(x) = 2x$ is integrable on the interval $[-2, 1]$ because it is continuous on $[-2, 1]$.
- Moreover, the definition of integrability implies that any partition whose norm approaches 0 can be used to determine the limit.
- For computational convenience, define Δ by subdividing $[-2, 1]$ into n subintervals of equal width

$$\Delta x_i = \Delta x = \frac{b-a}{n} = \frac{3}{n}.$$

- Choosing c_i as the right endpoint of each subinterval produces

$$c_i = a + i(\Delta x) = -2 + \frac{3i}{n}.$$

- So, the definite integral is given by

$$
\begin{aligned}
\int_{-2}^{1} 2x\,\mathrm{d}x &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} 2\left(-2+\frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} \left(-2+\frac{3i}{n}\right) = \lim_{n\to\infty} \frac{6}{n}\left\{-2n+\frac{3}{n}\left[\frac{n(n+1)}{2}\right]\right\} \\
&= \lim_{n\to\infty} \left(-12+9+\frac{9}{n}\right) = -3. \qquad \blacksquare
\end{aligned}
$$

- Because the definite integral in Example 2 is negative, it does not represent the area of the region shown in Figure 17.
- Definite integrals can be positive, negative, or zero. For a definite integral to be interpreted as an area, the function f must be continuous and nonnegative on $[a, b]$, as stated in the following theorem.

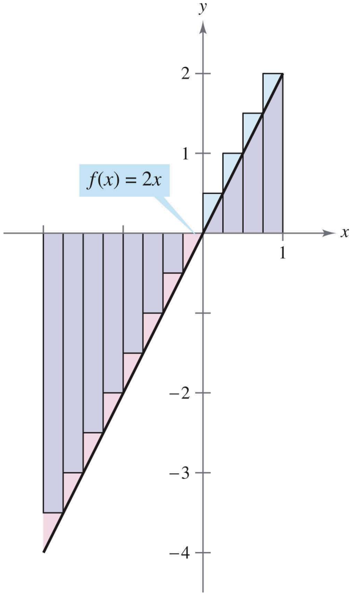


Figure 17: Because the definite integral is negative, it does not represent the area of the region.

Theorem 4.5 (The definite integral as the area of a region)

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by (See Figure 18)

$$\text{Area} = \int_a^b f(x)\,\mathrm{d}x.$$

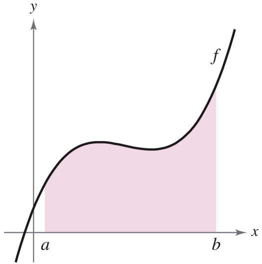


Figure 18: You can use a definite integral to find the area of the region bounded by the graph of f, the x-axis, $x = a$, and $x = b$.

- As an example of Theorem 4.6, consider the region bounded by the graph of $f(x) = 4x - x^2$ and the x-axis, as shown in Figure 19.

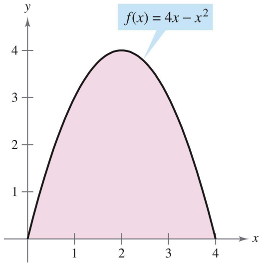


Figure 19: Area $= \int_0^4 (4x - x^2)\,\mathrm{d}x$.

- Because f is continuous and nonnegative on the closed interval $[0, 4]$, the area of the region is

$$\text{Area} = \int_0^4 (4x - x^2)\,\mathrm{d}x.$$

- You can evaluate a definite integral in two ways—you can use the limit definition or you can check to see whether the definite integral represents the area of a common geometric region such as a rectangle, triangle, or semicircle.

Example 3 (Areas of common geometric figures)

Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula.

a. $\int_1^3 4\,\mathrm{d}x$ **b.** $\int_0^3 (x+2)\,\mathrm{d}x$ **c.** $\int_{-2}^2 \sqrt{4-x^2}\,\mathrm{d}x$

a. This region is a rectangle of height 4 and width 2.

$$\int_1^3 4\,\mathrm{d}x = (\text{Area of rectangle}) = 4(2) = 8$$

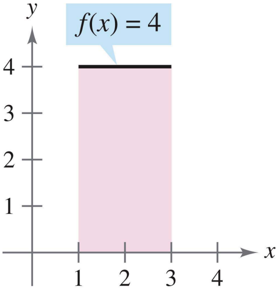


Figure 20: Area of region bounded by the graph of $f(x) = 4$, the x-axis, $x = 1$, and $x = 3$.

b. This region is a trapezoid with an altitude of 3 and parallel bases of lengths 2 and 5. The formula for the area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$.

$$\int_0^3 (x+2)\,\mathrm{d}x = (\text{Area of trapezoid}) = \frac{1}{2}(3)(2+5) = \frac{21}{2}$$

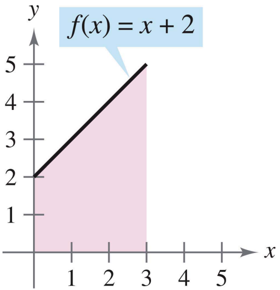


Figure 21: Area of region bounded by the graph of $f(x) = x + 2$, the x-axis, $x = 0$, and $x = 3$.

c. This region is a semicircle of radius 2. The formula for the area of a semicircle is $\frac{1}{2}\pi r^2$.

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = (\text{Area of semicircle}) = \frac{1}{2}\pi(2^2) = 2\pi$$

■

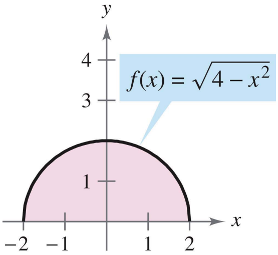


Figure 22: Area of region bounded by the graph of $f(x) = \sqrt{4 - x^2}$, the x-axis, $x = -2$, and $x = 2$.

- The variable of integration in a definite integral is sometimes called a dummy variable because it can be replaced by any other variable without changing the value of the integral. For instance, the definite integrals

$$\int_0^3 (x+2)\,\mathrm{d}x \quad \text{and} \quad \int_0^3 (t+2)\,\mathrm{d}t$$

  have the same value.

- The definition of the definite integral of f on the interval $[a, b]$ specifies that $a < b$.
- Now, however, it is convenient to extend the definition to cover cases in which $a = b$ or $a > b$.
- Geometrically, the following two definitions seem reasonable. For instance, it makes sense to define the area of a region of zero width and finite height to be 0.

Definition 4.6 (Two special definite integrals)

1. If f is defined at $x = a$, then we define $\int_a^a f(x)\,\mathrm{d}x = 0$.
2. If f is integrable on $[a, b]$, then we define $\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$.

Example 4 (Evaluating definite integrals)

Evaluate each definite integral. **a.** $\int_\pi^\pi \sin x\,\mathrm{d}x$ **b.** $\int_3^0 (x+2)\,\mathrm{d}x$

a. Because the sine function is defined at $x = \pi$, and the upper and lower limits of integration are equal, you can write

$$\int_\pi^\pi \sin x\,\mathrm{d}x = 0.$$

b. The integral $\int_0^3 (x+2)\,\mathrm{d}x$ has a value of $\frac{21}{2}$ you can write

$$\int_3^0 (x+2)\,\mathrm{d}x = -\int_0^3 (x+2)\,\mathrm{d}x = -\frac{21}{2}.$$

In Figure 24, the larger region can be divided at $x = c$ into two subregions whose intersection is a line segment. Because the line segment has zero area, it follows that the area of the larger region is equal to the sum of the areas of the two smaller regions.

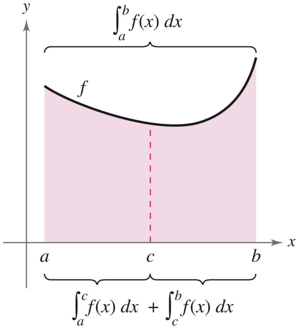


Figure 23: Additive interval property.

Theorem 4.6 (Additive interval property)

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x. \qquad \textit{See Figure24}$$

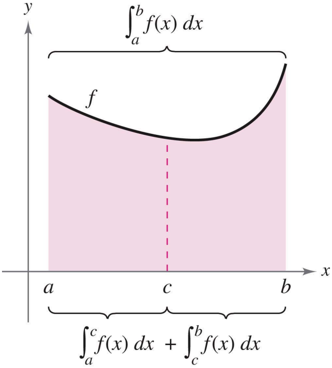


Figure 24: Additive interval property.

Example 5 (Using the additive interval property)

$$\int_{-1}^{1} |x| \, dx = \int_{-1}^{0} -x \, dx + \int_{0}^{1} x \, dx = \frac{1}{2} + \frac{1}{2} = 1 \qquad \blacksquare$$

Theorem 4.7 (Properties of definite integrals)

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx.$
2. $\int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx.$

- Note that Property 2 of Theorem 4.7 can be extended to cover any finite number of functions. For example,

$$\int_a^b [f(x) + g(x) + h(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx + \int_a^b h(x) \, dx.$$

Example 6 (Evaluation of a definite integral)

Evaluate $\int_1^3 (-x^2 + 4x - 3)\,\mathrm{d}x$ using each of the following values.

$$\int_1^3 x^2\,\mathrm{d}x = \frac{26}{3}, \quad \int_1^3 x\,\mathrm{d}x = 4, \quad \int_1^3 \mathrm{d}x = 2$$

$$\begin{aligned}
\int_1^3 (-x^2 + 4x - 3)\,\mathrm{d}x &= \int_1^3 (-x^2)\,\mathrm{d}x + \int_1^3 4x\,\mathrm{d}x + \int_1^3 (-3)\,\mathrm{d}x \\
&= -\int_1^3 x^2\,\mathrm{d}x + 4\int_1^3 x\,\mathrm{d}x - 3\int_1^3 \mathrm{d}x \\
&= -\left(\frac{26}{3}\right) + 4(4) - 3(2) = \frac{4}{3}
\end{aligned}$$

■

- If f and g are continuous on the closed interval $[a, b]$ and

$$0 \leq f(x) \leq g(x)$$

for $a \leq x \leq b$, the following properties are true.
  - First, the area of the region bounded by the graph of f and the x-axis (between a and b) must be nonnegative.
  - Second, this area must be less than or equal to the area of the region bounded by the graph of g and the x-axis (between a and b), as shown in Figure 25.
- These two properties are generalized in Theorem 4.8.

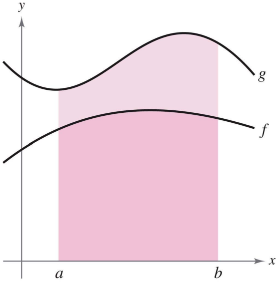


Figure 25: If $f(x) \leq g(x)$, then $\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$.

Theorem 4.8 (Preservation of inequality)

1. *If f is integrable and nonnegative on the closed interval $[a, b]$, then*

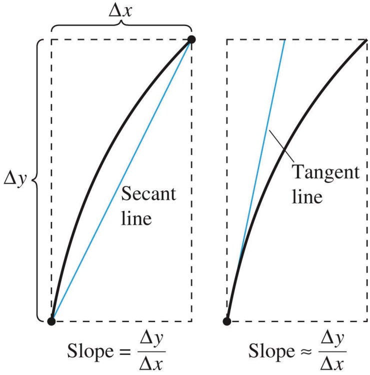
$$0 \leq \int_a^b f(x)\,\mathrm{d}x.$$

2. *If f and g are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$, then*

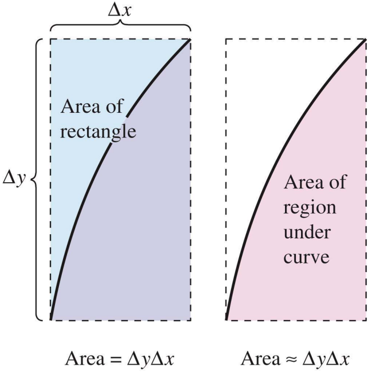
$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x.$$

Table of Contents

- The two major branches of calculus: differential calculus and integral calculus. The close connection was discovered independently by Isaac Newton and Gottfried Leibniz and is stated in a theorem that is appropriately called the Fundamental Theorem of Calculus.
- Informally, the theorem states that differentiation and (definite) integration are inverse operations, in the same sense that division and multiplication are inverse operations.



(a) Differentiation



(b) Definite integration

- The slope of the tangent line was defined using the quotient $\Delta y/\Delta x$.
- Similarly, the area of a region under a curve was defined using the product $\Delta y \Delta x$.
- So, at least in the primitive approximation stage, the operations of differentiation and definite integration appear to have an inverse relationship in the same sense that division and multiplication are inverse operations.
- The Fundamental Theorem of Calculus states that the limit processes preserve this inverse relationship.

Theorem 4.9 (The Fundamental Theorem of Calculus)

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

- The key to the proof is in writing the difference $F(b) - F(a)$ in a convenient form. Let Δ be any partition of $[a, b]$.

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

- By pairwise subtraction and addition of like terms, you can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) \\ &\quad - \cdots - F(x_1) + F(x_1) - F(x_0) \\ &= \sum_{i=1}^{n} \left[F(x_i) - F(x_{i-1})\right]. \end{aligned}$$

- By the Mean Value Theorem, you know that there exists a number c_i in the subinterval such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}.$$

- Because $F'(c_i) = f(c_i)$, you can let $\Delta x = x_i - x_{i-1}$ and obtain

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

- This important equation tells you that by repeatedly applying the Mean Value Theorem, you can always find a collection of c_i's such that the constant $F(b) - F(a)$ is a Riemann sum of f on $[a, b]$ for any partition.
- Theorem 4.4 guarantees that the limit of Riemann sums over the partition with $\|\Delta\| \to 0$ exists.
- So, taking the limit (as $\|\Delta\| \to 0$) produces

$$F(b) - F(a) = \int_a^b f(x)\,\mathrm{d}x.$$

Guidelines for using the Fundamental Theorem of Calculus

1. Provided you can find an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the following notation is convenient.
$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big]_a^b = F(b) - F(a)$$
For instance, to evaluate $\int_1^3 x^3\,\mathrm{d}x$, you can write
$$\int_1^3 x^3\,\mathrm{d}x = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$
3. It is not necessary to include a constant of integration C in the antiderivative because
$$\int_a^b f(x)\,\mathrm{d}x = [F(x) + C]_a^b = [F(b) + C] - [F(a) + C]$$
$$= F(b) - F(a).$$

Example 1 (Evaluating a definite integral)

Evaluate each definite integral.

a. $\int_1^2 (x^2 - 3)\,\mathrm{d}x$ **b.** $\int_1^4 3\sqrt{x}\,\mathrm{d}x$ **c.** $\int_0^{\pi/4} \sec^2 x\,\mathrm{d}x$

a. $\int_1^2 (x^2 - 3)\,\mathrm{d}x = \left[\frac{x^3}{3} - 3x\right]_1^2 = \left(\frac{8}{3} - 6\right) - \left(\frac{1}{3} - 3\right) = -\frac{2}{3}$

b. $\int_1^4 3\sqrt{x}\,\mathrm{d}x = 3\int_1^4 x^{1/2}\,\mathrm{d}x = 3\left[\frac{x^{3/2}}{3/2}\right]_1^4 = 2(4)^{3/2} - 2(1)^{3/2} = 14$

c. $\int_0^{\pi/4} \sec^2 x\,\mathrm{d}x = \tan x]_0^{\pi/4} = 1 - 0 = 1$ ■

Example 2 (Definite integral involving absolute value)

Evaluate $\int_0^2 |2x - 1|\,\mathrm{d}x$.

- Using Figure 27 and the definition of absolute value, you can rewrite the integrand as shown.

$$|2x - 1| = \begin{cases} -(2x - 1), & x < \frac{1}{2} \\ 2x - 1, & x \geq \frac{1}{2} \end{cases}$$

- From this, you can rewrite the integral in two parts.

$$\begin{aligned}\int_0^2 |2x-1|\,\mathrm{d}x &= \int_0^{1/2} -(2x-1)\,\mathrm{d}x + \int_{1/2}^2 (2x-1)\,\mathrm{d}x \\ &= \left[-x^2+x\right]_0^{1/2} + \left[x^2-x\right]_{1/2}^2 \\ &= \left(-\frac{1}{4}+\frac{1}{2}\right) - (0+0) + (4-2) - \left(\frac{1}{4}-\frac{1}{2}\right) = \frac{5}{2}\end{aligned}$$

■

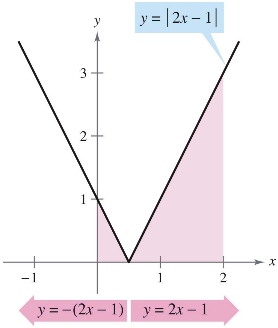


Figure 27: The definite integral of $y = |2x-1|$ on $[0, 2]$ is $\frac{5}{2}$.

Example 3 (Using the Fundamental Theorem to find area)

Find the area of the region bounded by the graph of $y = 2x^2 - 3x + 2$, the x-axis, and the vertical lines $x = 0$ and $x = 2$, as shown in Figure 28.

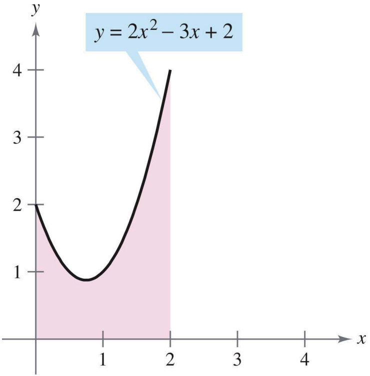


Figure 28: The area of the region bounded by the graph of $y = 2x^2 - 3x + 2$, the x-axis, $x = 0$, and $x = 2$ is $\frac{10}{3}$.

Note that $y > 0$ on the interval $[0, 2]$.

$$\int_0^2 (2x^2 - 3x + 2)\,\mathrm{d}x = \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^2$$
$$= \left(\frac{16}{3} - 6 + 4\right) - (0 - 0 + 0) = \frac{10}{3} \qquad \blacksquare$$

- The area of a region under a curve is greater than the area of an inscribed rectangle and less than the area of a circumscribed rectangle.
- The Mean Value Theorem for integrals states that somewhere "between" the inscribed and circumscribed rectangles there is a rectangle whose area is precisely equal to the area of the region under the curve, as shown in Figure 29.

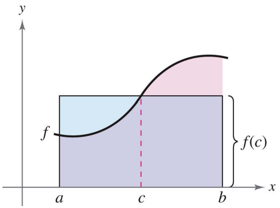


Figure 29: Mean value rectangle: $f(c)(b-a) = \int_a^b f(x)\,\mathrm{d}x$.

Theorem 4.10 (Mean Value Theorem for Integrals)

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,\mathrm{d}x = f(c)(b-a).$$

- Case 1: If f is constant on the interval $[a, b]$, the theorem is clearly valid because c can be any point in $[a, b]$.
- Case 2: If f is not constant on $[a, b]$, then, by the Extreme Value Theorem, you can choose $f(m)$ and $f(M)$ to be the minimum and maximum values of f on $[a, b]$.
- Because $f(m) \leq f(x) \leq f(M)$ for all x in $[a, b]$, you can apply Theorem 4.8 to write the following.

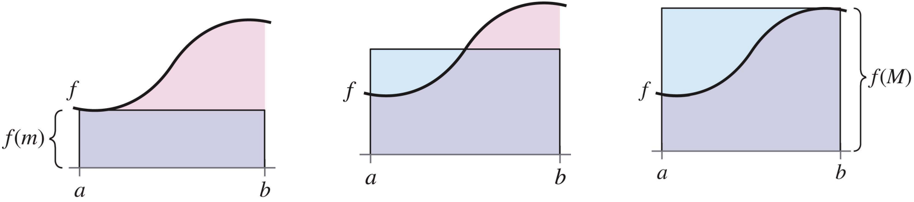
$$\int_a^b f(m)\,\mathrm{d}x \leq \int_a^b f(x)\,\mathrm{d}x \leq \int_a^b f(M)\,\mathrm{d}x \quad \text{See Figure 30}$$

$$f(m)(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq f(M)(b-a)$$

$$f(m) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq f(M)$$

- From the third inequality, you can apply the Intermediate Value Theorem to conclude that there exists some c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \quad \text{or} \quad f(c)(b-a) = \int_a^b f(x)\,\mathrm{d}x.$$



(a) Inscribed rectangle (less than actual area).

(b) Mean value rectangle (equal to actual area).

(c) Circumscribed rectangle (greater than actual area).

Figure 30: Mean Value Theorem for integrals.

- The value of $f(c)$ given in the Mean Value Theorem for integrals is called the average value of f on the interval $[a, b]$.

Definition 4.7 (The average value of a function on an interval)

If f is integrable on the closed interval $[a, b]$, then the average value of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x. \qquad \text{See Figure 31}$$

- In Figure 31 the area of the region under the graph of f is equal to the area of the rectangle whose height is the average value.
- To see why the average value of f is defined in this way, suppose that you partition $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$.
- If c_i is any point in the ith subinterval, the arithmetic average (or mean) of the function values at the c_i's is given by

$$a_n = \frac{1}{n}\left[f(c_1) + f(c_2) + \cdots + f(c_n)\right]. \quad \text{Average of } f(c_1), \ldots, f(c_n)$$

- By multiplying and dividing by $(b - a)$ you can write the average as

$$\begin{aligned} a_n &= \frac{1}{n}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{b-a}\right) \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{n}\right) = \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x. \end{aligned}$$

- Finally, taking the limit as $n \to \infty$ produces the average value of f on the interval $[a, b]$ as given in the definition above.

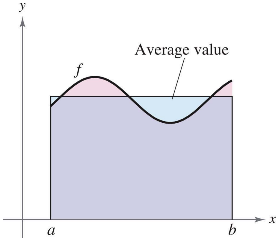


Figure 31: Average value $= \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$.

Example 4 (Finding the average value of a function)

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

The average value is given by (See Figure 32)

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = \frac{1}{4-1}\int_1^4 (3x^2 - 2x)\,\mathrm{d}x$$
$$= \frac{1}{3}\left[x^3 - x^2\right]_1^4 = \frac{1}{3}\left[64 - 16 - (1-1)\right] = \frac{48}{3} = 16.$$

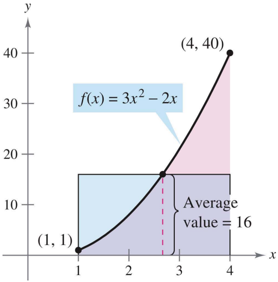


Figure 32: The average value of $f(x) = 3x^2 - 2x$, $1 < x < 4$.

- The definite integral of f on the interval $[a, b]$ is defined using the constant b as the upper limit of integration and x as the variable of integration.
- A slightly different situation may arise in which the variable x is used in the upper limit of integration.
- To avoid the confusion of using x in two different ways, t is temporarily used as the variable of integration.

The Definite Integral as a Number	The Definite Integral as a Function of x
$\int_a^b f(x)\,\mathrm{d}x$	$F(x) = \int_a^x f(t)\,\mathrm{d}t$
a: Constant, b: Constant, f: function of x	a: Constant, F: function of x, f: function of t

Example 6 (The definite integral as a function)

Evaluate the function
$$F(x) = \int_0^x \cos t\,\mathrm{d}t$$
at $x = 0$, $\pi/6$, $\pi/4$, $\pi/3$ and $\pi/2$.

- You could evaluate five different definite integrals, one for each of the given upper limits.
- However, it is much simpler to fix x (as a constant) temporarily to obtain
$$\int_0^x \cos t\,\mathrm{d}t = \sin t]_0^x = \sin x - \sin 0 = \sin x.$$

- Now, using $F(x) = \sin x$, you can obtain the results shown in Figure 33. ■

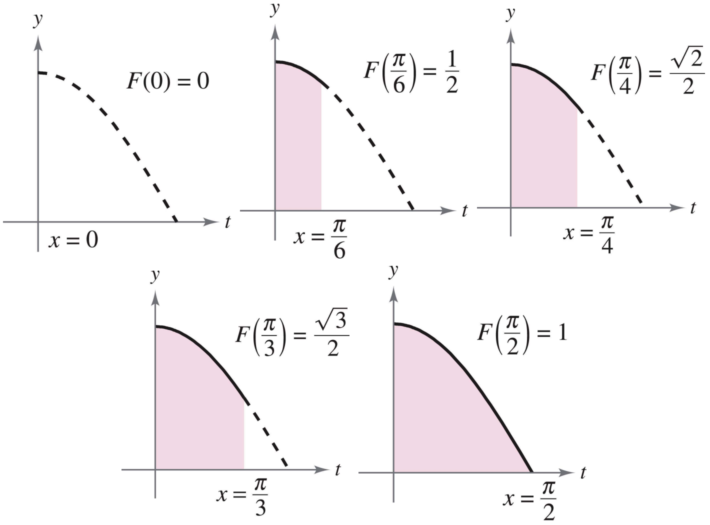


Figure 33: $F(x) = \int_0^x \cos t \, dt$ is the area under the curve $f(t) = \cos t$ from 0 to x.

- The function $F(x)$ as accumulating the area under the curve $f(t) = \cos t$ from $t = 0$ to $t = x$.
- For $x = 0$, the area is 0 and $F(0) = 0$. For $x = \pi/2$, $F(\pi/2) = 1$ gives the accumulated area under the cosine curve on the entire interval $[0, \pi/2]$.
- This interpretation of an integral as an accumulation function is used often in applications of integration.
- The derivative of F is the original integrand. That is,

$$\frac{\mathrm{d}}{\mathrm{d}x}[F(x)] = \frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^x \cos t \,\mathrm{d}t\right] = \cos x.$$

- This result is generalized in the following theorem, called the Second Fundamental Theorem of Calculus.

Theorem 4.11 (The Second Fundamental Theorem of Calculus)

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(t)\,\mathrm{d}t\right] = f(x).$$

- Begin by defining F as

$$F(x) = \int_a^x f(t)\,\mathrm{d}t.$$

- Then, by the definition of the derivative, you can write

$$\begin{aligned}
F'(x) &= \lim_{\Delta x\to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t\right] \\
&= \lim_{\Delta x\to 0} \frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)\,\mathrm{d}t + \int_x^a f(t)\,\mathrm{d}t\right] \\
&= \lim_{\Delta x\to 0} \frac{1}{\Delta x}\left[\int_x^{x+\Delta x} f(t)\,\mathrm{d}t\right].
\end{aligned}$$

- From the Mean Value Theorem for Integrals (assuming $\Delta x > 0$), you know there exists a number c in the interval $[x, x + \Delta x]$ such that the integral in the expression above is equal to $f(c)\Delta x$.
- Moreover, because $x \leq c \leq x + \Delta x$, it follows that $c \to x$ as $\Delta x \to 0$.
- So, you obtain

$$F'(x) = \lim_{\Delta x \to 0} \left[\frac{1}{\Delta x} f(c)\Delta x \right] = \lim_{\Delta x \to 0} f(c) = f(x).$$

- A similar argument can be made for $\Delta x < 0$. □

- Using the area model for definite integrals, you can view the approximation

$$f(x)\Delta x \approx \int_{x}^{x+\Delta x} f(t)\,\mathrm{d}t$$

as saying that the area of the rectangle of height $f(x)$ and width Δx is approximately equal to the area of the region lying between the graph of f and the x-axis on the interval $[x, x+\Delta x]$, as shown in Figure 34.

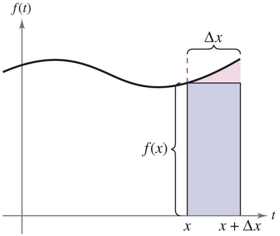


Figure 34: $f(x)\Delta x \approx \int_{x}^{x+\Delta x} f(t)\,\mathrm{d}t$.

Example 7 (Using the Second Fundamental Theorem of Calculus)

Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^x \sqrt{t^2+1}\,\mathrm{d}t\right]$.

- Note that $f(t) = \sqrt{t^2+1}$ is continuous on the entire real line.
- So, using the Second Fundamental Theorem of Calculus, you can write

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^x \sqrt{t^2+1}\,\mathrm{d}t\right] = \sqrt{x^2+1}.$$

■

Example 8 (Using the Second Fundamental Theorem of Calculus)

Find the derivative $F(x) = \int_{\pi/2}^{x^3} \cos t\,\mathrm{d}t$.

Using $u = x^3$ you can apply the Second Fundamental Theorem of Calculus with the Chain Rule as shown

$$F'(x) = \frac{\mathrm{d}F}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}u}[F(x)]\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$= \frac{\mathrm{d}}{\mathrm{d}u}\left[\int_{\pi/2}^{x^3} \cos t \, \mathrm{d}t\right] \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}u}\left[\int_{\pi/2}^{u} \cos t \, \mathrm{d}t\right] \frac{\mathrm{d}u}{\mathrm{d}x}$$

$$= (\cos u)(3x^2) = (\cos x^3)(3x^2).$$

■

- The Fundamental Theorem of Calculus states that if f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) \, \mathrm{d}x = F(b) - F(a).$$

- But because $F'(x) = f(x)$, this statement can be rewritten as

$$\int_a^b F'(x) \, \mathrm{d}x = F(b) - F(a)$$

where the quantity $F(b) - F(a)$ represents the net change of F on the interval $[a, b]$.

Theorem 4.12 (The Net Change Theorem)

The definite integral of the rate of change of a quantity $F'(x)$ gives the total change, or net change, in that quantity on the interval $[a, b]$.

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a) \qquad \textit{Net change of } F(x)$$

Example 9 (Using the Net Change Theorem)

A chemical flows into a storage tank at a rate of $180 + 3t$ liters per minute, where $0 \le t \le 60$. Find the amount of the chemical that flows into the tank during the first 20 minutes.

- Let $c(t)$ be the amount of the chemical in the tank at time t.
- Then $c'(t)$ represents the rate at which the chemical flows into the tank at time t.

- During the first 20 minutes, the amount that flows into the tank is

$$\int_0^{20} c'(t)\,\mathrm{d}t = \int_0^{20} (180+3t)\,\mathrm{d}t$$
$$= \left[180t + \frac{3}{2}t^2\right]_0^{20} = 3600 + 600 = 4200.$$

- So, the amount that flows into the tank during the first 20 minutes is 4200 liters. ∎
- The velocity of a particle moving along a straight line where $s(t)$ is the position at time t. Then its velocity is $v(t) = s'(t)$ and

$$\int_a^b v(t)\,\mathrm{d}t = s(b) - s(a).$$

- This definite integral represents the net change in position, or displacement, of the particle.
- When calculating the total distance traveled by the particle, you must consider the intervals where $v(t) \le 0$ and the intervals where $v(t) \ge 0$.

- When $v(t) \leq 0$ the particle moves to the left, and when $v(t) \geq 0$, the particle moves to the right. To calculate the total distance traveled, integrate the absolute value of velocity $|v(t)|$.
- So, the displacement of a particle and the total distance traveled by a particle over $[a, b]$ can be written as (see Figure 35)

$$\text{Displacement on } [a, b] = \int_a^b v(t)\,\mathrm{d}t = A_1 - A_2 + A_3$$

Total distance traveled on $[a, b]$

$$= \int_a^b |v(t)|\,\mathrm{d}t = A_1 + A_2 + A_3.$$

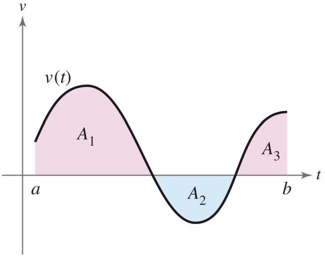


Figure 35: A_1, A_2, and A_3 are the areas of the shaded regions.

Example 10 (Solving a particle motion problem)

A particle is moving along a line so that its velocity is $v(t) = t^3 - 10t^2 + 29t - 20$ meters per second at time t.

a. What is the displacement of the particle on the time interval $1 \le t \le 5$?

b. What is the total distance traveled by the particle on the time interval $1 \le t \le 5$?

a. By definition, you know that the displacement is

$$\begin{aligned}\int_1^5 v(t)\,\mathrm{d}t &= \int_1^5 (t^3 - 10t^2 + 29t - 20)\,\mathrm{d}t \\ &= \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_1^5 = \frac{25}{12} - \left(-\frac{103}{12}\right) = \frac{128}{12} \\ &= \frac{32}{3}.\end{aligned}$$

So, the particle moves $\frac{32}{3}$ meters to the right.

b. To find the total distance traveled, calculate $\int_1^5 |v(t)|\,\mathrm{d}t$. Using the fact that $v(t)$ can be factored as $(t-1)(t-4)(t-5)$, you can determine that $v(t) \geq 0$ on $[1,4]$ and on $v(t) \leq 0$ on $[4,5]$. So, the total distance traveled is

$$\begin{aligned}
\int_1^5 |v(t)|\,\mathrm{d}t &= \int_1^4 v(t)\,\mathrm{d}t - \int_4^5 v(t)\,\mathrm{d}t \\
&= \int_1^4 (t^3 - 10t^2 + 29t - 20)\,\mathrm{d}t \\
&\quad - \int_4^5 (t^3 - 10t^2 + 29t - 20)\,\mathrm{d}t \\
&= \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_1^4 \\
&\quad - \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_4^5 \\
&= \frac{45}{4} - \left(-\frac{7}{12}\right) = \frac{71}{6} \text{ meters.}
\end{aligned}$$

■

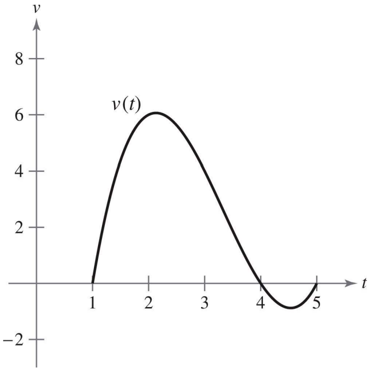


Figure 36: Total distance traveled.

Table of Contents

- In this section you will study techniques for integrating composite functions.
- The discussion is split into two parts—pattern recognition and change of variables. Both techniques involve a u-substitution.
- With pattern recognition you perform the substitution mentally, and with change of variables you write the substitution steps.
- The role of substitution in integration is comparable to the role of the Chain Rule in differentiation.
- Recall that for differentiable functions given by $y = F(u)$ and $u = g(x)$, the Chain Rule states that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[F(g(x))\right] = F'(g(x))g'(x).$$

- From the definition of an antiderivative, it follows that

$$\int F'(g(x))g'(x)\,\mathrm{d}x = F(g(x)) + C.$$

- These results are summarized in the following theorem.

Theorem 4.13 (Antidifferentiation of a composite function)

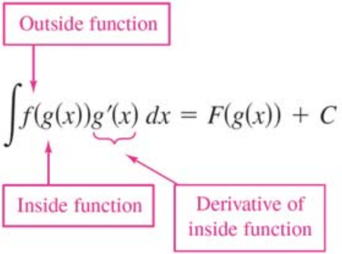
Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,\mathrm{d}x = F(g(x)) + C.$$

Letting $u = g(x)$ gives $\mathrm{d}u = g'(x)\,\mathrm{d}x$ and

$$\int f(u)\,\mathrm{d}u = F(u) + C.$$

- Example 1 and 2 show how to apply Theorem 4.13 directly, by recognizing the presence of $f(g(x))$ and $g'(x)$.
- Note that the composite function in the integrand has an outside function f and an inside function g.
- Moreover, the derivative $g'(x)$ is present as a factor of the integrand.



Example 1 (Recognizing the $f(g(x))g'(x)$ pattern)

Find $\int (x^2+1)^2(2x)\,\mathrm{d}x$.

- Letting $g(x) = x^2 + 1$, you obtain
$$g'(x) = 2x$$
and
$$f(g(x)) = f(x^2+1) = (x^2+1)^2.$$
- From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern.

- Using the Power Rule for Integration and Theorem 4.13, you can write

$$\int \overbrace{(x^2+1)^2}^{f(g(x))} \overbrace{(2x)}^{g'(x)} \, \mathrm{d}x = \frac{1}{3}(x^2+1)^3 + C.$$

- Try using the Chain Rule to check that the derivative of $\frac{1}{3}(x^2+1)^3 + C$ is the integrand of the original integrand. ■

Example 2 (Recognizing the $f(g(x))g'(x)$ pattern)

Find $\int 5 \cos 5x \, \mathrm{d}x$.

- Letting $g(x) = 5x$, you obtain

$$g'(x) = 5$$

and

$$f(g(x)) = f(5x) = \cos 5x.$$

- From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern.

- Using the Cosine Rule for Integration and Theorem 4.13, you can write

$$\int \overbrace{(\cos 5x)}^{f(g(x))} \overbrace{(5)}^{g'(x)} \, \mathrm{d}x = \sin 5x + C.$$

- You can check this by differentiating $\sin 5x + C$ to obtain the original integrand. ■
- The integrands in Example 1 and Example 2 fit the $f(g(x))g'(x)$ pattern exactly—you only had to recognize the pattern.
- You can extend this technique considerably with the Constant Multiple Rule

$$\int kf(x) \, \mathrm{d}x = k \int f(x) \, \mathrm{d}x.$$

- Many integrands contain the essential part (the variable part) of $g'(x)$ but are missing a constant multiple.
- In such cases, you can multiply and divide by the necessary constant multiple, as shown in Example 3.

Example 3 (Multiplying and dividing by a constant)

Find $\int x(x^2+1)^2\,\mathrm{d}x$.

- This is similar to the integral given in Example 1, except that the integrand is missing a factor of 2.
- Recognizing that $2x$ is the derivative of x^2+1, you can let $g(x)=x^2+1$ and supply the $2x$ as follows.

$$\int x(x^2+1)^2\,\mathrm{d}x = \int (x^2+1)^2\left(\frac{1}{2}\right)(2x)\,\mathrm{d}x = \frac{1}{2}\int \overbrace{(x^2+1)^2}^{f(g(x))}\overbrace{(2x)}^{g'(x)}\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\frac{(x^2+1)^3}{3}\right] + C = \frac{1}{6}(x^2+1)^3 + C. \qquad \blacksquare$$

Change of variables for indefinite integrals

- With a formal change of variables, you completely rewrite the integral in terms of u and $\mathrm{d}u$ (or any other convenient variable).
- Although this procedure can involve more written steps than the pattern recognition illustrated in Examples 1 and 3, it is useful for complicated integrands.
- The change of variables technique uses the Leibniz notation for the differential. That is, if $u = g(x)$, then $\mathrm{d}u = g'(x)\,\mathrm{d}x$, and the integral in Theorem 4.13 takes the form

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u = F(u) + C.$$

Example 4 (Change of variables)

Find $\int \sqrt{2x-1}\,\mathrm{d}x$.

- First, let u be the inner function, $u = 2x - 1$.
- Then calculate the differential $\mathrm{d}u$ to be $\mathrm{d}u = 2\,\mathrm{d}x$.
- Now, using $\sqrt{2x-1} = \sqrt{u}$ and $\mathrm{d}x = \mathrm{d}u/2$ substitute to obtain

$$\int \sqrt{2x-1}\,\mathrm{d}x = \int \sqrt{u}\left(\frac{\mathrm{d}u}{2}\right) = \frac{1}{2}\int u^{1/2}\,\mathrm{d}u$$

$$= \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C = \frac{1}{3}\,u^{3/2} + C = \frac{1}{3}(2x-1)^{3/2} + C.$$

■

Example 5 (Change of variables)

Find $\int x\sqrt{2x-1}\,\mathrm{d}x$.

As in the previous example, let $u = 2x - 1$ and obtain $\mathrm{d}x = \mathrm{d}u/2$. Because the integrand contains a factor of x, you must also solve for x in terms of u, as show

$$u = 2x - 1 \quad \Longrightarrow \quad x = (u+1)/2.$$

Now, using substitution, you obtain

$$\begin{aligned}\int x\sqrt{2x-1}\,\mathrm{d}x &= \int \left(\frac{u+1}{2}\right) u^{1/2} \left(\frac{\mathrm{d}u}{2}\right) = \frac{1}{4}\int (u^{3/2} + u^{1/2})\,\mathrm{d}u \\ &= \frac{1}{4}\left(\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C \\ &= \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C.\end{aligned}$$

■

Example 6 (Change of variables)

Find $\int \sin^2 3x \cos 3x \, \mathrm{d}x$.

- Let $u = \sin 3x$. Then $\mathrm{d}u = (\cos 3x)(3) \, \mathrm{d}x$.
- Now, $\frac{\mathrm{d}u}{3} = \cos 3x \, \mathrm{d}x$ substitute to obtain

$$\begin{aligned} \int \sin^2 3x \cos 3x \, \mathrm{d}x &= \int u^2 \left(\frac{\mathrm{d}u}{3}\right) = \frac{1}{3} \int u^2 \, \mathrm{d}u \\ &= \frac{1}{3} \left(\frac{u^3}{3}\right) + C = \frac{1}{9} \sin^3 3x + C. \end{aligned}$$

■

The steps used for integration by substitution are summarized in the following guidelines.

Guidelines for making a change of variables

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the inner part of a composite function, such as a quantity raised to a power.
2. Compute $\mathrm{d}u = g'(x)\,\mathrm{d}x$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

The General Power Rule for integration

- One of the most common u-substitutions involves quantities in the integrand that are raised to a power.
- Because of the importance of this type of substitution, it is given a special name—the General Power Rule for Integration.

Theorem 4.14 (The General Power Rule for Integration)

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\,\mathrm{d}x = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\,\mathrm{d}u = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

Example 7 (Substitution and the General Power Rule)

a. $\int 3(3x-1)^4\,\mathrm{d}x = \int \overbrace{(3x-1)^4}^{u^4}\overbrace{(3)\,\mathrm{d}x}^{\mathrm{d}u} = \dfrac{\overbrace{(3x-1)^5}^{u^5/5}}{5} + C$

b. $\int (2x+1)(x^2+x)\,\mathrm{d}x = \int \overbrace{(x^2+x)^1}^{u^1}\overbrace{(2x+1)\,\mathrm{d}x}^{\mathrm{d}u} = \dfrac{\overbrace{(x^2+x)^2}^{u^2/2}}{2} + C$

c. $\int 3x^2\sqrt{x^3-2}\,\mathrm{d}x = \int \overbrace{(x^3-2)^{1/2}}^{u^{1/2}}\overbrace{(3x^2)\,\mathrm{d}x}^{\mathrm{d}u} = \dfrac{\overbrace{(x^3-2)^{3/2}}^{u^{3/2}/(3/2)}}{3/2} + C =$
$\frac{2}{3}(x^3-2)^{3/2} + C$

d. $\int \frac{-4x}{(1-2x^2)^2}\,\mathrm{d}x = \int \overbrace{(1-2x^2)^{-2}}^{u^{-2}}\overbrace{(-4x)\,\mathrm{d}x}^{\mathrm{d}u} = \dfrac{\overbrace{(1-2x^2)^{-1}}^{u^{-1}/(-1)}}{-1} + C$
$= -\frac{1}{1-2x^2} + C$

e. $\int \cos^2 x \sin x\,\mathrm{d}x = -\int \overbrace{(\cos x)^2}^{u^2}\overbrace{(-\sin x)\,\mathrm{d}x}^{\mathrm{d}u} = -\dfrac{\overbrace{(\cos x)^3}^{u^3/3}}{3} + C$ ■

Change of variables for definite integrals

- When using u-substitution with a definite integral, it is often convenient to determine the limits of integration for the variable u rather than to convert the antiderivative back to the variable x and evaluate at the original limits.
- This change of variables is stated explicitly in the next theorem.

Theorem 4.15 (Change of variables for definite integrals)

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u.$$

Example 8 (Change of variables)

Evaluate $\int_0^1 x(x^2+1)^3\,\mathrm{d}x$.

- To evaluate this integral, let $u = x^2 + 1$. Then, you obtain

$$u = x^2 + 1 \quad \Longrightarrow \quad \mathrm{d}u = 2x\,\mathrm{d}x.$$

- Before substituting, determine the new upper and lower limits of integration.

Lower Limit	Upper Limit
When $x = 0$, $u = 0^2 + 1 = 1$.	When $x = 1$, $u = 1^2 + 1 = 2$.

- Now, you can substitute to obtain

$$\begin{aligned}\int_0^1 x(x^2+1)^3\,\mathrm{d}x &= \frac{1}{2}\int_0^1 (x^2+1)^3(2x)\,\mathrm{d}x = \frac{1}{2}\int_1^2 u^3\,\mathrm{d}u \\ &= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 = \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}.\end{aligned}$$

- Try rewriting the antiderivative $\frac{1}{2}(u^4/4)$ in terms of the variable x and evaluate the definite integral at the original limits of integration, as shown.

$$\frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 = \frac{1}{2}\left[\frac{(x^2+1)^4}{4}\right]_0^1 = \frac{1}{2}\left(4-\frac{1}{4}\right) = \frac{15}{8}$$

Notice that you obtain the same result. ■

Example 9 (Change of variables)

Evaluate $\int_1^5 \frac{x}{\sqrt{2x-1}}\,\mathrm{d}x$.

- To evaluate this integral, let $u = \sqrt{2x-1}$.
- Then, you obtain

$$u^2 = 2x-1 \qquad u^2+1 = 2x \qquad \frac{u^2+1}{2} = x \qquad u\,\mathrm{d}u = \mathrm{d}x.$$

- Before substituting, determine the new upper and lower limits of integration.

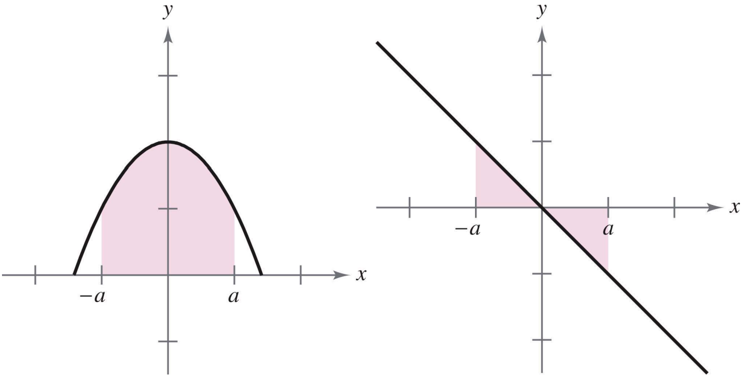
Lower Limit	Upper Limit
When $x = 1$, $u = \sqrt{2-1} = 1$.	When $x = 5$, $u = \sqrt{10-1} = 3$.

- Now, substitute to obtain

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,\mathrm{d}x = \int_1^3 \frac{1}{u}\left(\frac{u^2+1}{2}\right) u\,\mathrm{d}u = \frac{1}{2}\int_1^3 (u^2+1)\,\mathrm{d}u$$

$$= \frac{1}{2}\left[\frac{u^3}{3} + u\right]_1^3 = \frac{1}{2}\left(9 + 3 - \frac{1}{3} - 1\right) = \frac{16}{3}. \quad \blacksquare$$

- Even with a change of variables, integration can be difficult.
- Occasionally, you can simplify the evaluation of a definite integral over an interval that is symmetric about the y-axis or about the origin by recognizing the integrand to be an even or odd function (see Figure 37).



(a) Even function (b) Odd function

Figure 37: Even and odd functions.

Theorem 4.16 (Integration of even and odd functions)

Let f be integrable on the closed interval $[-a, a]$.

1. *If f is an even function, then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x$.*
2. *If f is an odd function, then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$.*

- Because f is even, you know that $f(x) = f(-x)$. Using Theorem 4.13 with the substitution $u = -x$ produces

$$\int_{-a}^{0} f(x)\,\mathrm{d}x = \int_{a}^{0} f(-u)(-\,\mathrm{d}u) = -\int_{a}^{0} f(u)\,\mathrm{d}u$$
$$= \int_{0}^{a} f(u)\,\mathrm{d}u = \int_{0}^{a} f(x)\,\mathrm{d}x.$$

- Finally, using Theorem 4.6, you obtain

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x$$
$$= \int_{0}^{a} f(x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x.$$

- This proves the first property. The proof of the second property is left to you (see Exercise 101). □

Example 10 (Integration of an odd function)

Evaluate $\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\,\mathrm{d}x$.

- Letting $f(x) = \sin^3 x \cos x + \sin x \cos x$ produces

$$\begin{aligned} f(-x) &= \sin^3(-x)\cos(-x) + \sin(-x)\cos(-x) \\ &= -\sin^3 x \cos x - \sin x \cos x = -f(x). \end{aligned}$$

- So, f is an odd function, and because f is symmetric about the origin over $[-\pi/2, \pi/2]$, you can apply Theorem 4.16 to conclude that

$$\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\,\mathrm{d}x = 0.$$

■

- From Figure 38, we can see that the two regions on either side of the y-axis have the same area.

- However, because one lies below the x-axis and one lies above it, integration produces a cancellation effect. (More will be said about areas below the x-axis in Section 7.1.)

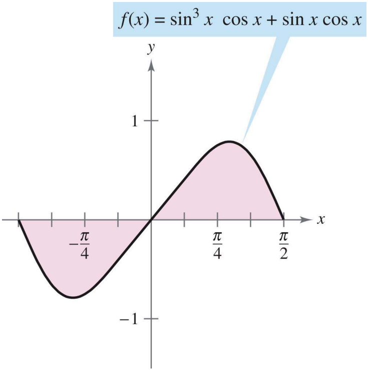


Figure 38: Integration of an odd function $f(x) = \sin^3 x \cos x + \sin x \cos x$, $-\pi/2 < x < \pi/2$.