Chapter 3 Applications of Differentiation

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Extrema of a function

- In calculus, much effort is devoted to determining the behavior of a function f on an interval I.
- Does f have a maximum value on I? Does it have a minimum value? Where is the function increasing? Where is it decreasing?
- In this chapter you will learn how derivatives can be used to answer these questions.

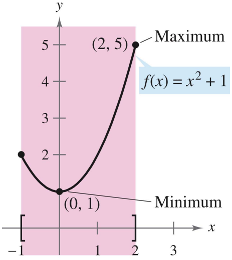
Definition 3.1 (Extrema)

Let f be defined on an interval I containing c

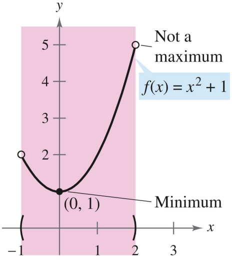
1. $f(c)$ is the minimum of f on I if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the maximum of f on I if $f(c) \geq f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the extreme values, or extrema (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the absolute minimum and absolute maximum, or the global minimum and global maximum, on the interval.

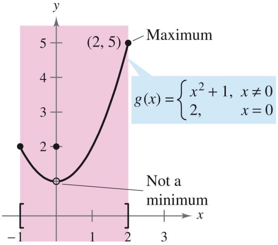
- A function need not have a minimum or a maximum on an interval. For instance, in Figure 1 (a) and 1 (b), you can see that the function $f(x) = x^2 + 1$ has both a minimum and a maximum on the closed interval $[-1, 2]$, but does not have a maximum on the open interval $(-1, 2)$. Moreover, in Figure 1 (c), you can see that continuity (or the lack of it) can affect the existence of an extremum on the interval.



(a) f is continuous, $[-1, 2]$ is closed.



(b) f is continuous, $(-1, 2)$ is open.



(c) g is not continuous, $[-1, 2]$ is closed.

Figure 1: Extrema can occur at interior points or endpoints of an interval. Extrema that occur at the endpoints are called endpoint extrema.

Theorem 3.1 (The Extreme Value Theorem)

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

Definition 3.2 (Relative extrema)

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a relative maximum of f, or you can say that f has a relative maximum at $(c, f(c))$.
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a relative minimum of f, or you can say that f has a relative minimum at $(c, f(c))$.

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima. Relative maximum and relative minimum are sometimes called local maximum and local minimum, respectively.

- In Figure 2, the graph of $f(x) = x^3 - 3x^2$ has a relative maximum at the point $(0, 0)$ and a relative minimum at the point $(2, -4)$.
- Informally, for a continuous function, you can think of a relative maximum as occurring on a "hill" on the graph, and a relative minimum as occurring in a "valley" on the graph. Such a hill and valley can occur in two ways.
  - If the hill (or valley) is smooth and rounded, the graph has a horizontal tangent line at the high point (or low point).
  - If the hill (or valley) is sharp and peaked, the graph represents a function that is not differentiable at the high point (or low point).

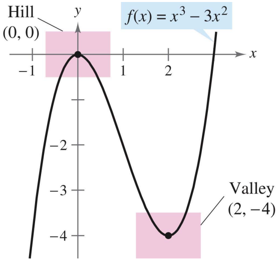
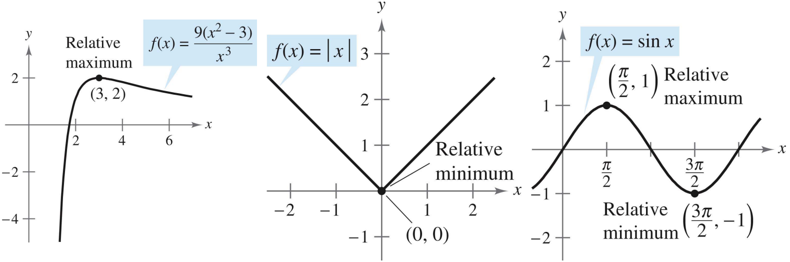


Figure 2: f has a relative maximum at $(0, 0)$ and a relative minimum at $(2, -4)$.

Example 1 (The value of the derivative at relative extrema)

Find the value of the derivative at each relative extremum shown in Figure 3.



(a) $f'(3) = 0$ (b) $f'(0)$ does not exist (c) $f'\left(\frac{\pi}{2}\right) = 0; f'\left(\frac{3\pi}{2}\right) = 0$

Figure 3: The value of the derivative at relative extrema.

(a) The derivative of $f(x) = \frac{9(x^2-3)}{x^3} = 9(x^{-1} - 3x^{-3})$ is

$$f'(x) = \frac{x^3(18x) - (9)(x^2-3)(3x^2)}{(x^3)^2} = \frac{9(9-x^2)}{x^4}$$

$$f'(x) = (9(x^{-1} - 3x^{-3}))' = 9\left(\frac{-1}{x^2} + \frac{9}{x^4}\right) = \frac{9(9-x^2)}{x^4}.$$

At the point $(3, 2)$, the value of the derivative is $f'(3) = 0$.

(b) At $x = 0$, the derivative of $f(x) = |x|$ does not exist because the following one-sided limits differ.

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$$

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{x}{x} = 1$$

(c) The derivative of $f(x) = \sin x$ is $f'(x) = \cos x$.
At the point $(\pi/2, 1)$, the value of the derivative is
$f'(\pi/2) = \cos(\pi/2) = 0$.
At the point $(3\pi/2, -1)$, the value of the derivative is
$f'(3\pi/2) = \cos(3\pi/2) = 0$.

- Note in Example 1 that at each relative extremum, the derivative either is zero or does not exist. The x-values at these special points are called critical numbers.
- Figure 4 illustrates the two types of critical numbers. Notice in the definition that the critical number c has to be in the domain of f, but c does not have to be in the domain of f'.

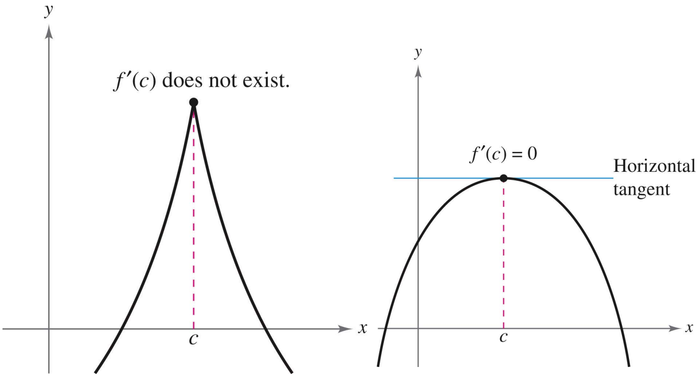


Figure 4: c is a critical number of f.

Definition 3.3 (Critical number)

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a critical number of f.

Theorem 3.2 (Relative extrema occur only at critical numbers)

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

- Case 1: If f is not differentiable at $x = c$, then, by definition, c is a critical number of f and the theorem is valid.
- Case 2: If f is differentiable at $x = c$, then $f'(c)$ must be positive, negative, or 0.

  Suppose $f'(c)$ is positive. Then

  $$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0$$

  which implies that there exists an interval (a, b) containing c such that

$$\frac{f(x)-f(c)}{x-c} > 0, \quad \text{for all } x \neq c \text{ in } (a,b).$$

[See Exercise 84(b), Section 1.2.]

- Because this quotient is positive, the signs of the denominator and numerator must agree. This produces the following inequalities for x-values in the interval (a, b).
  - Left of c: $x < c$ and $f(x) < f(c) \implies f(c)$ is not a relative minimum
  - Right of c: $x > c$ and $f(x) > f(c) \implies f(c)$ is not a relative maximum
- So, the assumption that $f'(c) > 0$ contradicts the hypothesis that $f(c)$ is a relative extremum. Assuming that $f'(c) < 0$ produces a similar contradiction, you are left with only one possibility-namely, $f'(c) = 0$. So, by definition, c is a critical number of f and the theorem is valid. $\square$

Finding extrema on a closed interval

Guidelines for finding extrema on a closed interval

1. Find the critical numbers of f in (a, b).
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each endpoint of $[a, b]$.
4. The least of these values is the minimum. The greatest is the maximum.

Example 2 (Finding extrema on a closed interval)

Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$.

- Begin by differentiating the function.

$$f(x) = 3x^4 - 4x^3 \qquad f'(x) = 12x^3 - 12x^2$$

- To find the critical numbers of f, you must find all x-values for which $f'(x) = 0$ and all x-values for which $f'(x)$ does not exist.

$$f'(x) = 12x^3 - 12x^2 = 0 \qquad 12x^2(x-1) = 0 \qquad x = 0,\ 1$$

- Because f' is defined for all x, you can conclude that these are the only critical numbers of f. By evaluating f at these two critical numbers and at the endpoints of $[-1, 2]$, you can determine that the maximum is $f(2) = 16$ and the minimum is $f(1) = -1$, as shown in the table.

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = 7$	$f(0) = 0$	$f(1) = -1$ Minimum	$f(2) = 16$ Maximum

- The graph of f is shown in Figure 5. ■

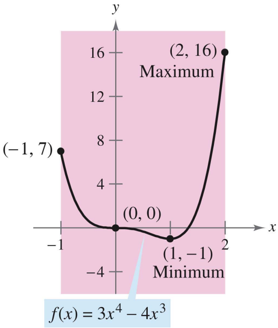


Figure 5: On the closed interval $[-1, 2]$, $f(x) = 3x^4 - 4x^3$ has a minimum at $(1, -1)$ and a maximum at $(2, 16)$.

- In Figure 5, note that the critical number $x = 0$ does not yield a relative minimum or a relative maximum. This tells you that the converse of Theorem 3.2 is not true.
- In other words, the critical numbers of a function need not produce relative extrema.

Example 3 (Finding extrema on a closed interval)

Find the extrema of $f(x) = 2x - 3x^{2/3}$ on the interval $[-1, 3]$.

- Begin by differentiating the function.

$$f(x) = 2x - 3x^{2/3} \qquad f'(x) = 2 - \frac{2}{x^{1/3}} = 2\left(\frac{x^{1/3} - 1}{x^{1/3}}\right)$$

- From this derivative, you can see that the function has two critical numbers in the interval $(-1, 3)$. The number 1 is a critical number because $f'(1) = 0$, and the number 0 is a critical number because $f'(0)$ does not exist. By evaluating f at these two numbers and at the endpoints of the interval, you can conclude that the minimum is $f(-1) = -5$ and the maximum is $f(0) = 0$, as shown in the table. ■

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = -5$ Minimum	$f(0) = 0$ Maximum	$f(1) = -1$	$f(3) = 6 - 3\sqrt[3]{9}$ ≈ -0.24

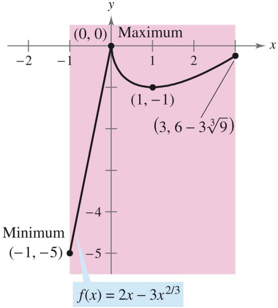


Figure 6: On the closed interval $[-1,3]$, $f(x)=2x-3x^{2/3}$ has a minimum at $(-1,-5)$ and a maximum at $(0,0)$.

Example 4 (Finding extrema on a closed interval)

Find the extrema of $f(x)=2\sin x-\cos 2x$ on the interval $[0,2\pi]$.

- This function is differentiable for all real x, so you can find all critical numbers by differentiating the function and setting $f'(x)$ equal to zero, as shown.

$$
\begin{aligned}
f(x) &= 2\sin x - \cos 2x \\
f'(x) &= 2\cos x + 2\sin 2x = 0 \\
2\cos x + 4\cos x \sin x &= 0 \\
2(\cos x)(1 + 2\sin x) &= 0
\end{aligned}
$$

- In the interval $[0, 2\pi]$, the factor $\cos x$ is zero when $x = \pi/2$ and when $x = 3\pi/2$. The factor $(1 + 2\sin x)$ is zero when $x = 7\pi/6$ and when $x = 11\pi/6$.
- By evaluating f at these four critical numbers and at the endpoints of the interval, you can conclude that the maximum is $f(\pi/2) = 3$ and the minimum occurs at two points, $f(7\pi/6) = -3/2$ and $f(11\pi/6) = -3/2$, as shown in the table.
- The graph is shown in Figure 7. ■

Left Endpoint	Critical Number	Critical Number	Critical Number	Critical Number	Right Endpoint
$f(0) = -1$	$f\left(\frac{\pi}{2}\right) = 3$ Maximum	$f\left(\frac{7\pi}{6}\right) = -\frac{3}{2}$ Minimum	$f\left(\frac{3\pi}{2}\right) = -1$	$f\left(\frac{11\pi}{6}\right) = -\frac{3}{2}$ Minimum	$f(2\pi) = -1$

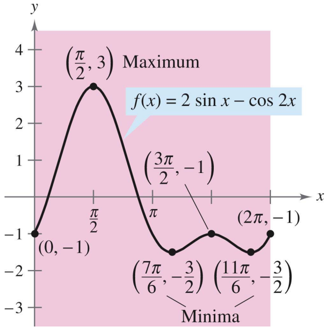


Figure 7: On the closed interval $[0, 2\pi]$, f has two minima at $(7\pi/6, -3/2)$ and $(11\pi/6, -3/2)$ and a maximum at $(\pi/2, 3)$.

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Rolle's Theorem

- The Extreme Value Theorem (Section 1) states that a continuous function on a closed interval $[a, b]$ must have both a minimum and a maximum on the interval. Both of these values, however, can occur at the endpoints
- Rolle's Theorem, named after the French mathematician Rolle, Michel, gives conditions that guarantee the existence of an extreme value in the <u>interior of a closed interval.</u>
- A theorem of calculus that ensures the existence of a critical point between any two points on a "nice" function that have the same y-value.

Theorem 3.3 (Rolle's Theorem)

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If

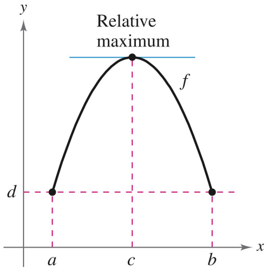
$$f(a) = f(b)$$

then there is at least one number c in (a, b) such that $f'(c) = 0$.

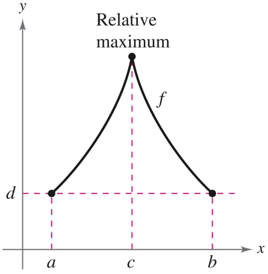
Let $f(a) = d = f(b)$.

- Case 1: If $f(x) = d$ for all x in $[a, b]$, f is constant on the interval and, by Theorem 2.2, $f'(x) = 0$ for all x in (a, b).
- Case 2: Suppose $f(x) > d$ for some x in (a, b).
  - By the Extreme Value Theorem, you know that f has a maximum at some c in the interval.
  - Moreover, because $f(c) > d$, this maximum does not occur at either endpoint. So, f has a maximum in the open interval (a, b).
  - This implies that $f(c)$ is a relative maximum and, by Theorem 3.2, c is a critical number of f.
  - Finally, because f is differentiable at c, you can conclude that $f'(c) = 0$.

- Case 3: If $f(x) < d$ for some x in (a, b), you can use an argument similar to Case 2, but involving minimum instead of the maximum. $\square$
- From Rolle's Theorem, if a function f is continuous on $[a, b]$ and differentiable on (a, b), and if $f(a) = f(b)$, there must be at least one x-value between a and b at which the graph of f has a horizontal tangent. If the differentiability requirement is dropped from Rolle's Theorem, f will still have a critical number in (a, b), but it may not yield a horizontal tangent. Such a case is shown below.



(a) f is continuous on $[a, b]$ and differentiable on (a, b).



(b) f is continuous on $[a, b]$ but not differentiable on (a, b).

Example 1 (Illustrating Rolle's Theorem)

Find the two x-intercepts of

$$f(x) = x^2 - 3x + 2$$

and show that $f'(x) = 0$ at some point between the two x-intercepts.

- Note that f is differentiable on the entire real line. Setting $f(x)$ equal to 0 produces

$$x^2 - 3x + 2 = 0 \qquad (x-1)(x-2) = 0$$

- So, $f(1) = f(2) = 0$, and from Rolle's Theorem you know that there exists at least one c in the interval $(1, 2)$ such that $f'(c) = 0$.
- To find such a c, you can solve the equation

$$f'(x) = 2x - 3 = 0$$

and determine that $f'(x) = 0$ when $x = \frac{3}{2}$.
- Note that this x-value lies in the open interval $(1, 2)$, as shown in Figure 9.

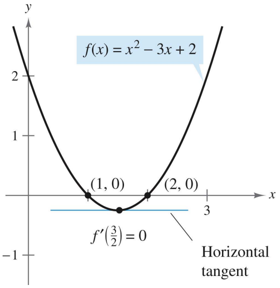


Figure 9: The x-value for which $f'(x) = 0$ is between the two x-intercepts.

Example 2 (Illustrating Rolle's Theorem)

Let $f(x) = x^4 - 2x^2$. Find all values of c in the interval $(-2, 2)$ such that $f'(c) = 0$.

- To begin, note that the function satisfies the conditions of Rolle's Theorem. That is, f is continuous on the interval $[-2, 2]$ and differentiable on the interval $(-2, 2)$.

- Moreover, because $f(-2) = f(2) = 8$, you can conclude that there exists at least one c in $(-2, 2)$ such that $f'(c) = 0$.
- Setting the derivative equal to 0 produces

$$f'(x) = 4x^3 - 4x = 0 \quad 4x(x-1)(x+1) = 0 \quad x = 0,\ 1,\ -1.$$

- The graph of f is shown in Figure 10. ■

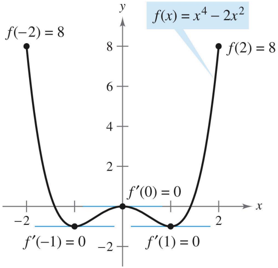


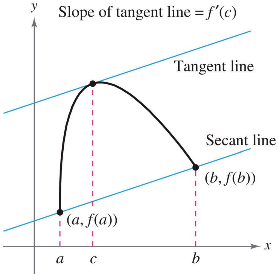
Figure 10: $f'(x) = 0$ for more than one x-value in the interval $(-2, 2)$.

- Rolle's Theorem can be used to prove the Mean Value Theorem. A major theorem of calculus that relates values of a function to a value of its derivative. Essentially the theorem states that for a "nice" function, there is a tangent line parallel to any secant line.

Theorem 3.4 (The Mean Value Theorem)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



- Refer to the above figure. The equation of the secant line that passes through the points $(a, f(a))$ and $(b, f(b))$ is
$$y = \left[\frac{f(b) - f(a)}{b - a}\right](x - a) + f(a).$$
- Let $g(x)$ be the difference between $f(x)$ and y. Then
$$g(x) = f(x) - y = f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a) - f(a).$$
- By evaluating g at a and b, you can see that $g(a) = 0 = g(b)$. Because f is continuous on $[a, b]$, it follows that g is also continuous on $[a, b]$. Furthermore, because f is differentiable, g is also differentiable, and you can apply Rolle's Theorem to the function g.
- So, there exists a number c in (a, b) such that $g'(c) = 0$, thus
$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$
- So, there exists a number c in (a, b) such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- Although the Mean Value Theorem can be used directly in problem solving, it is used more often to prove other theorems.
- In fact, some people consider this to be the most important theorem in calculus-it is closely related to the Fundamental Theorem of Calculus discussed in Section 4.4.
- Geometrically, the theorem guarantees the existence of a tangent line that is parallel to the secant line through the points $(a, f(a))$ and $(b, f(b))$.
- In terms of rates of change, the Mean Value Theorem implies that there must be a point in the open interval (a, b) at which the instantaneous rate of change is equal to the average rate of change over the interval $[a, b]$.

Example 3 (Finding a tangent line)

Given $f(x) = 5 - (4/x)$, find all values of c in the open interval $(1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

- The slope of the secant line through $(1, f(1))$ and $(4, f(4))$ is

$$\frac{f(4) - f(1)}{4 - 1} = \frac{4 - 1}{4 - 1} = 1$$

- Note that the function satisfies the conditions of the Mean Value Theorem. That is, f is continuous on the interval $[1, 4]$ and differentiable on the interval $(1, 4)$. So, there exists at least one number c in $(1, 4)$ such that $f'(c) = 1$.
- Solving the equation $f'(x) = 1$ yields

$$f'(x) = \frac{4}{x^2} = 1$$

which implies that $x = \pm 2$.

- So, in the interval $(1, 4)$, you can conclude that $c = 2$, as shown in Figure 11. ■

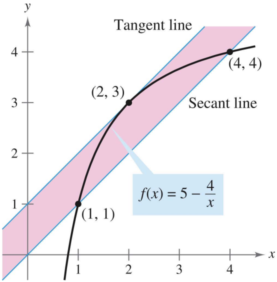


Figure 11: The tangent line at $(2, 3)$ is parallel to the secant line through $(1, 1)$ and $(4, 4)$.

- A useful alternative form of the Mean Value Theorem is as follows: If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number c in (a, b) such that alternative form of Mean Value Theorem

$$f(b) = f(a) + (b - a)f'(c).$$

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Definition 3.4 (Increasing and decreasing functions)

A function f is increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
A function f is decreasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

- A function is increasing if, as x moves to the right, its graph moves up, and is decreasing if its graph moves down. For example, the function in Figure 12 is decreasing on the interval $(-\infty, a)$ is constant on the interval (a, b) and is increasing on the interval (b, ∞).

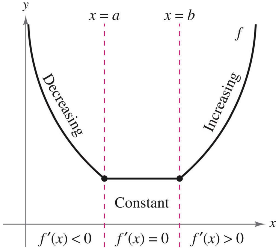


Figure 12: The derivative is related to the slope of a function.

Theorem 3.5 (Test for increasing and decreasing functions)

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

1. *If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.*
2. *If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.*
3. *If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.*

- To prove the first case, assume that $f'(x) > 0$ for all x in the interval (a, b) and let $x_1 < x_2$ be any two points in the interval.
- By the Mean Value Theorem, you know that there exists a number c such that $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

- Because $f'(c) > 0$ and $x_2 - x_1 > 0$, you know that

$$f(x_2) - f(x_1) > 0$$

which implies that $f(x_1) < f(x_2)$. So, f is increasing on the interval.

- The second case has a similar proof (see Exercise 97), and the third case is a consequence of Exercise 78 in Section 3.2. □

Example 1 (Intervals on which f is increasing or decreasing)

Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

- Note that f is differentiable on the entire real number line.
- To determine the critical numbers of f, set $f'(x)$ equal to zero.

$$f(x) = x^3 - \frac{3}{2}x^2 \quad f'(x) = 3x^2 - 3x = 0 \quad 3x(x-1) = 0 \quad x = 0,\ 1$$

- Because there are no points for which f' does not exist, you can conclude that $x = 0$ and $x = 1$ are the only critical numbers.
- The table summarizes the testing of the three intervals determined by these two critical numbers.

Interval	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test Value	$x = -1$	$x = \frac{1}{2}$	$x = 2$
Sign of $f'(x)$	$f'(-1) = 6 > 0$	$f'\left(\frac{1}{2}\right) = -\frac{3}{4} < 0$	$f'(2) = 6 > 0$
Conclusion	Increasing	Decreasing	Increasing

- So, f is increasing on the intervals $(-\infty, 0)$ and $(1, \infty)$ and decreasing on the interval $(0, 1)$, as shown in Figure 13. ■

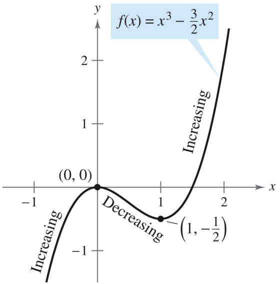


Figure 13: Increasing and decreasing intervals of $f(x) = x^3 - \frac{3}{2}x^2$.

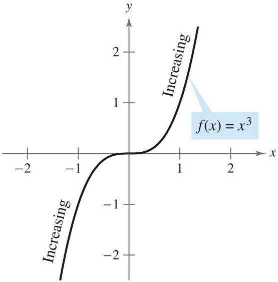
Guidelines for finding intervals on which a function is increasing or decreasing

Let f be continuous on the interval (a, b). To find the open intervals on which f is increasing or decreasing, use the following steps.

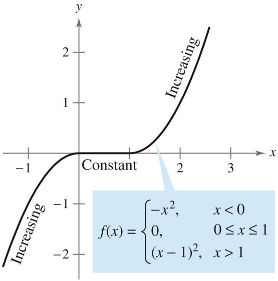
1. Locate the critical numbers of f in (a, b), and use these numbers to determine test intervals.
2. Determine the sign of $f'(x)$ at one test value in each of the intervals.
3. Use Theorem 3.5 to determine whether f is increasing or decreasing on each interval.

These guidelines are also valid if the interval (a, b) is replaced by an interval of the form $(-\infty, b)$, (a, ∞), or $(-\infty, \infty)$.

- A function is strictly monotonic on an interval if it is either increasing on the entire interval or decreasing on the entire interval.
- For instance, the function $f(x) = x^3$ is strictly monotonic on the entire real number line because it is increasing on the entire real number line, as shown in Figure 38. The function shown in the right panel of Figure 38 is not strictly monotonic on the entire real number line because it is constant on the interval $[0, 1]$.



(a) Strictly monotonic function.



(b) Not strictly monotonic function.

The First Derivative Test

- The intervals on which a function is increasing or decreasing, it is not difficult to locate the relative extrema of the function.
- For instance, in Figure 15 (from Example 1), the function

$$f(x) = x^3 - \frac{3}{2}x^2$$

  has a relative maximum at the point $(0, 0)$ because f is increasing immediately to the left of $x = 0$ and decreasing immediately to the right of $x = 0$.
- Similarly, f has a relative minimum at the point $\left(1, -\frac{1}{2}\right)$ because f is decreasing immediately to the left of $x = 1$ and increasing immediately to the right of $x = 1$.

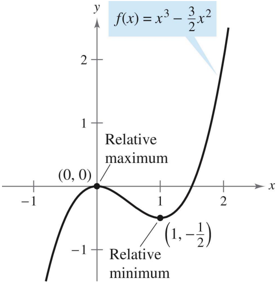


Figure 15: Relative extrema of $f(x) = x^3 - 3x^2/2$.

Theorem 3.6 (The First Derivative Test)

Let c a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.

1. *If $f'(x)$ changes from negative to positive at c, then f has a relative minimum at $(c, f(c))$.*
2. *If $f'(x)$ changes from positive to negative at c, then f has a relative maximum at $(c, f(c))$.*
3. *If $f'(x)$ is positive on both sides of c or negative on both sides of c, then f is neither a relative minimum nor a relative maximum.*

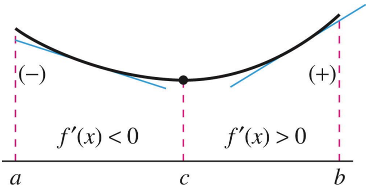
- Assume that $f'(x)$ changes from negative to positive at c. Then there exist a and b in I such that

$$f'(x) < 0 \quad \text{for all } x \text{ in } (a, c)$$

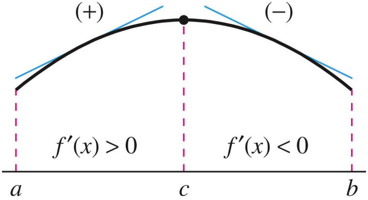
and

$$f'(x) > 0 \quad \text{for all } x \text{ in } (c, b).$$

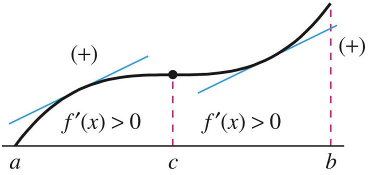
- By Theorem 3.5, f is decreasing on $[a, c]$ and increasing on $[c, b]$. So, $f(c)$ is a minimum of f on the open interval (a, b) and, consequently, a relative minimum of f.
- This proves the first case of the theorem. The second case can be proved in a similar way (see Exercise 98). $\square$



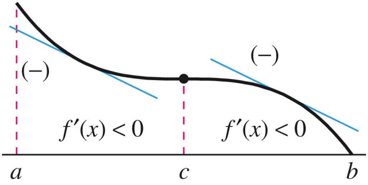
(a) Relative minimum.



(b) Relative maximum.



(c) Neither relative minimum nor relative maximum.



(d) Neither relative minimum nor relative maximum.

Example 2 (Applying the First Derivative Test)

Find the relative extrema of the function $f(x) = \frac{1}{2}x - \sin x$ in the interval $(0, 2\pi)$.

- Note that f is continuous on the interval $(0, 2\pi)$. To determine the critical numbers of f in this interval, set $f'(x)$ equal to 0.

$$f'(x) = \frac{1}{2} - \cos x = 0 \qquad \cos x = \frac{1}{2} \qquad x = \frac{\pi}{3}, \ \frac{5\pi}{3}$$

- Because there are no points for which f' does not exist, you can conclude that $x = \pi/3$ and $x = 5\pi/3$ are the only critical numbers.
- The table summarizes the testing of the three intervals determined by these two critical numbers.

Interval	$0 < x < \frac{\pi}{3}$	$\frac{\pi}{3} < x < \frac{5\pi}{3}$	$\frac{5\pi}{3} < x < 2\pi$
Test Value	$x = \frac{\pi}{4}$	$x = \pi$	$x = \frac{7\pi}{4}$
Sign of $f'(x)$	$f'\left(\frac{\pi}{4}\right) < 0$	$f'(\pi) > 0$	$f'\left(\frac{7\pi}{4}\right) < 0$
Conclusion	Decreasing	Increasing	Decreasing

- By applying the First Derivative Test, you can conclude that f has a relative minimum at the point where

$$x = \frac{\pi}{3} \qquad x\text{-value where relative minimum occurs}$$

and a relative maximum at the point where

$$x = \frac{5\pi}{3} \qquad x\text{-value where relative maximum occurs}$$

as shown in Figure 17. ■

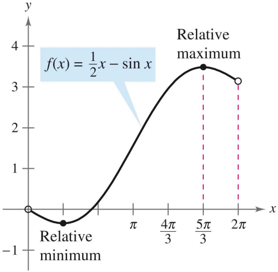


Figure 17: A relative minimum occurs where f changes from decreasing to increasing, and a relative maximum occurs where f changes from increasing to decreasing.

Example 3 (Applying the First Derivative Test)

Find the relative extrema of

$$f(x) = (x^2 - 4)^{2/3}.$$

- Begin by noting that f is continuous on the entire real number line.
- The derivative of f

$$f'(x) = \frac{2}{3}(x^2 - 4)^{-1/3}(2x) = \frac{4x}{3(x^2 - 4)^{1/3}}$$

  is 0 when $x = 0$ and does not exist when $x = \pm 2$. So, the critical numbers are $x = -2$, $x = 0$, and $x = 2$.
- The table summarizes the testing of the four intervals determined by these three critical numbers.
- By applying the First Derivative Test, you can conclude that f has a relative minimum at the point $(-2, 0)$, a relative maximum at the point $(0, \sqrt[3]{16})$, and another relative minimum at the point $(2, 0)$, as shown in Figure 18. ■

Interval	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test Value	$x = -3$	$x = -1$	$x = 1$	$x = 3$
Sign of $f'(x)$	$f'(-3) < 0$	$f'(-1) > 0$	$f'(1) < 0$	$f'(3) > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

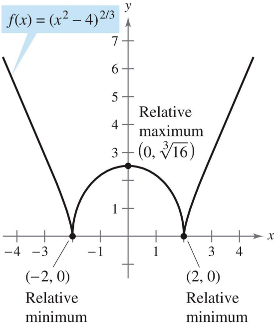


Figure 18: You can apply the First Derivative Test to find relative extrema.

Example 4 (Applying the First Derivative Test)

Find the relative extrema of

$$f(x) = \frac{x^4+1}{x^2}.$$

$$f(x) = x^2 + x^{-2}$$

$$f'(x) = 2x - 2x^{-3} = 2x - \frac{2}{x^3} = \frac{2(x^4-1)}{x^3} = \frac{2(x^2+1)(x-1)(x+1)}{x^3}$$

- So, $f'(x)$ is zero at $x = \pm 1$.
- Moreover, because $x = 0$ is not in the domain of f, you should use this x-value along with the critical numbers to determine the test intervals.

$$x = \pm 1 \qquad\qquad x = 0$$

- The table summarizes the testing of the four intervals determined by these three x-values.

Interval	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test Value	$x = -2$	$x = -\frac{1}{2}$	$x = \frac{1}{2}$	$x = 2$
Sign of $f'(x)$	$f'(-2) < 0$	$f'\left(-\frac{1}{2}\right) > 0$	$f'\left(\frac{1}{2}\right) < 0$	$f'(2) > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

- By applying the First Derivative Test, you can conclude that f has one relative minimum at the point $(-1, 2)$ and another at the point $(1, 2)$, as shown in Figure 19. ■

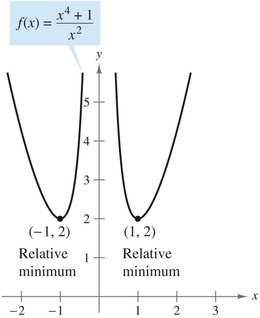


Figure 19: x-values that are not in the domain of f, as well as critical numbers, determine test intervals for f'.

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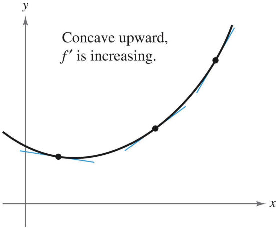
Definition 3.5 (Concavity)

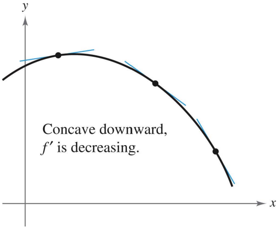
Let f be differentiable on an open interval I. The graph of f is concave upward on I if f' is increasing on the interval and concave downward on I if f' is decreasing on the interval.

The following graphical interpretation of concavity is useful.

1. Let f be differentiable on an open interval I. If the graph of f is concave upward on I, then the graph of f lies above all of its tangent lines on I. [See Figure 20 (a).]
2. Let f be differentiable on an open interval I. If the graph of f is concave downward on I, then the graph of f lies below all of its tangent lines on I. [See Figure 20 (b).]



(a) Concave upward: The graph of f lies above its tangent lines.



(b) Concave downward: The graph of f lies below its tangent lines.

Figure 20: Concave upward and downward.

Theorem 3.7 (Test for concavity)

Let f be a function whose second derivative exists on an open interval I.

1. *If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.*
2. *If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.*

Example 1 (Determining concavity)

Determine the open intervals on which the graph of

$$f(x) = \frac{6}{x^2+3}$$

is concave upward or downward.

- Begin by observing that f is continuous on the entire real line.
- Next, find the second derivative of f.

$$f(x) = 6(x^2+3)^{-1}$$

$$f'(x) = (-6)(x^2+3)^{-2}(2x) = \frac{-12x}{(x^2+3)^2}$$

$$f''(x) = \frac{(x^2+3)^2(-12) - (-12x)(2)(x^2+3)(2x)}{(x^2+3)^4} = \frac{36(x^2-1)}{(x^2+3)^3}$$

- Because $f''(x) = 0$ when $x = \pm 1$ and f'' is defined on the entire line, you should test f'' in the intervals $(-\infty, -1)$, $(1, -1)$, and $(1, \infty)$.
- The results are shown in the table and in Figure 21. ■

Interval	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Test Value	$x = -2$	$x = 0$	$x = 2$
Sign of $f''(x)$	$f''(-2) > 0$	$f''(0) < 0$	$f''(2) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

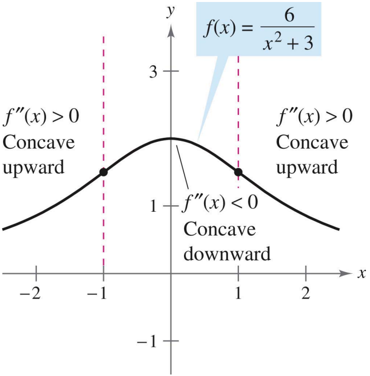


Figure 21: From the sign of f'' you can determine the concavity of the graph of $f(x) = \frac{6}{x^2+3}$.

Example 2 (Determining concavity)

Determine the open intervals on which the graph of $f(x) = \frac{x^2+1}{x^2-4}$ is concave upward or concave downward.

- Differentiating twice produces the following.

$$f(x) = \frac{x^2+1}{x^2-4} = 1 + \frac{5}{x^2-4}$$

$$f'(x) = \frac{(x^2-4)(2x) - (x^2+1)(2x)}{(x^2-4)^2} = \frac{-10x}{(x^2-4)^2}$$

$$f''(x) = \frac{(x^2-4)^2(-10) - (-10x)(2)(x^2-4)(2x)}{(x^2-4)^4} = \frac{10(3x^2+4)}{(x^2-4)^3}$$

- There are no points at which $f''(x) = 0$, but at $x = \pm 2$ the function f is not continuous, so test for concavity in the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$, as shown in the table.
- The graph of f is shown in Figure 22. ■

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Test Value	$x = -3$	$x = 0$	$x = 3$
Sign of $f''(x)$	$f''(-3) > 0$	$f''(0) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

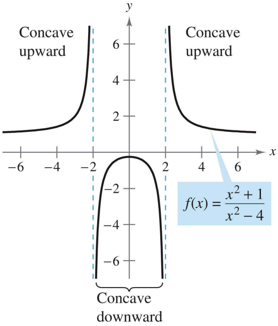


Figure 22: Determining concavity of $f(x) = \frac{x^2+1}{x^2-4}$.

- If the tangent line to the graph exists at such a point where the concavity changes, that point is a point of inflection.
- Three types of points of inflection are shown below.

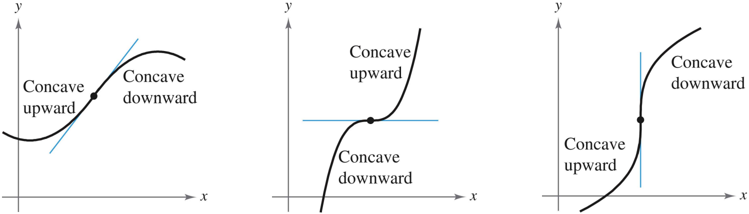


Figure 23: The concavity of changes at a point of inflection.
Note that the graph crosses its tangent line at point of inflection.

Definition 3.6 (Point of inflection)

Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a point of inflection of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point.

Theorem 3.8 (Points of inflection)

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f'' does not exist at $x = c$.

Example 3 (Finding points of inflection)

Determine the points of inflection and discuss the concavity of the graph of $f(x) = x^4 - 4x^3$.

- Differentiating twice produces the following.

$$f(x) = x^4 - 4x^3 \quad f'(x) = 4x^3 - 12x^2 \quad f''(x) = 12x^2 - 24x = 12x(x -$$

- Setting $f''(x) = 0$, you can determine that the possible points of inflection occur at $x = 0$ and $x = 2$.
- By testing the intervals determined by these x-values, you can conclude that they both yield points of inflection. A summary of this testing is shown in the table, and the graph of f is shown in Figure 24.

Interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test Value	$x = -1$	$x = 1$	$x = 3$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(1) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

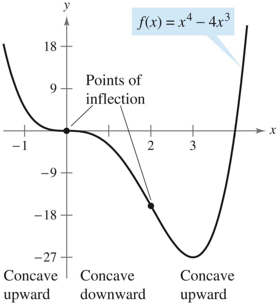


Figure 24: Points of inflection can occur where $f''(x) = 0$ or f'' does not exist.

The Second Derivative Test

A method for determining whether a critical point is a relative minimum or maximum.

Theorem 3.9 (The Second Derivative Test)

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. *If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.*
2. *If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.*

If $f''(c) = 0$, the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

- If $f'(c) = 0$ and $f''(c) > 0$, there exists an open interval I containing c for which
$$\frac{f'(x) - f'(c)}{x - c} = \frac{f'(x)}{x - c} > 0$$
for all $x \neq c$ in I.
- If $x < c$, then $x - c < 0$ and $f'(x) < 0$. Also, if $x > c$, then $x - c > 0$ and $f'(x) > 0$.
- So, $f'(x)$ changes from negative to positive at c, and the First Derivative Test implies that $f(c)$ is a relative minimum.
- A proof of the second case is left to you. $\square$

Example 4 (Using the Second Derivative Test)

Find the relative extrema for $f(x) = -3x^5 + 5x^3$.

- Begin by finding the critical numbers of f.

$$f'(x) = -15x^4 + 15x^2 = 15x^2(1 - x^2) = 0 \qquad x = -1,\ 0,\ 1$$

- Using

$$f''(x) = -60x^3 + 30x = 30(-2x^3 + x)$$

you can apply the Second Derivative Test as shown below.

Point	$(-1, -2)$	$(1, 2)$	$(0, 0)$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(1) < 0$	$f''(0) = 0$
Conclusion	Relative minimum	Relative maximum	Test fails

- Because the Second Derivative Test fails at $(0, 0)$, you can use the First Derivative Test and observe that f increases to the left and right of $x = 0$. So, $(0, 0)$ is neither a relative minimum nor a relative maximum (even though the graph has a horizontal tangent line at this point). ■

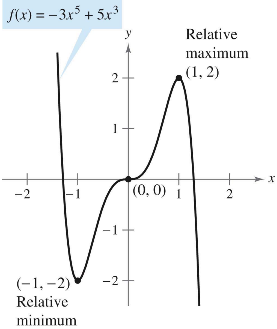


Figure 25: $(0, 0)$ is neither a relative minimum nor a relative maximum of $f(x) = -3x^5 + 5x^3$.

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- This section discusses the "end behavior" of a function on an infinite interval. Consider the graph of $f(x) = \frac{3x^2}{x^2+1}$ as shown in Figure 26. Graphically, you can see that the values of $f(x)$ appear to approach 3 as x increases without bound or decreases without bound.

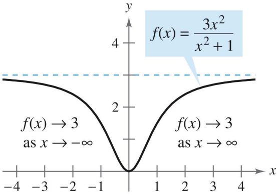


Figure 26: The limit of $f(x) = \frac{3x^2}{x^2+1}$ as x approaches $-\infty$ or ∞ is 3.

- You can come to the same conclusions numerically, as shown below.

x decreases without bound. | x increases without bound.

x	$-\infty \leftarrow$	-100	-10	-1	0	1	10	100	$\to \infty$
$f(x)$	$3 \leftarrow$	2.9997	2.97	1.5	0	1.5	2.97	2.9997	$\to 3$

$f(x)$ approaches 3. | $f(x)$ approaches 3.

- The table suggests that the value of $f(x)$ approaches 3 as x increases without bound ($x \to \infty$). Similarly, $f(x)$ approaches 3 as x decreases without bound ($x \to -\infty$).
- These limits at infinity are denoted by

$$\lim_{x \to -\infty} f(x) = 3 \qquad \text{Limit at negative infinity}$$

and

$$\lim_{x \to \infty} f(x) = 3. \qquad \text{Limit at positive infinity}$$

- To say that a statement is true as x increases without bound means that for some (large) real number M, the statement is true for all x in the interval $\{x : x > M\}$.
- The following definition uses this concept.

Definition 3.7 (Limits at infinity)

Let L be a real number.

1. The statement $\lim_{x\to\infty} f(x) = L$ means that for each $\epsilon > 0$ there exists an $M > 0$ such that $|f(x) - L| < \epsilon$ whenever $x > M$.
2. The statement $\lim_{x\to-\infty} f(x) = L$ means that for each $\epsilon > 0$ there exists an $N < 0$ such that $|f(x) - L| < \epsilon$ whenever $x < N$.

- The definition is shown below. Note that for a given positive number ϵ there exists a positive number M such that, for $x > M$, the graph of f will lie between the horizontal lines given by $y = L + \epsilon$ and $y = L - \epsilon$.

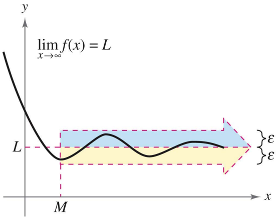


Figure 27: $f(x)$ is within ϵ units of L as $x \to \infty$.

Horizontal asymptotes

- In Figure 27, the graph of f approaches the line $y = L$ as x increases without bound.
- The line $y = L$ is called a horizontal asymptote of the graph of f.

Definition 3.8 (Horizontal asymptote)

The line $y = L$ is a horizontal asymptote of the graph of f if

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L.$$

- For example, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ both exist, then

$$\lim_{x\to\infty}[f(x)+g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x).$$

and

$$\lim_{x\to\infty}[f(x)g(x)] = \left[\lim_{x\to\infty} f(x)\right]\left[\lim_{x\to\infty} g(x)\right].$$

Similar properties hold for limits at $-\infty$.

Theorem 3.10 (Limits at infinity)

If r is a positive rational number and c is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0.$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0.$$

Example 1 (Finding a limit at infinity)

Find the limit:

$$\lim_{x\to\infty} \left(5 - \frac{2}{x^2}\right).$$

- Using Theorem 3.10, you can write

$$\lim_{x\to\infty} \left(5 - \frac{2}{x^2}\right) = \lim_{x\to\infty} 5 - \lim_{x\to\infty} \frac{2}{x^2} = 5 - 0 = 5.$$

- The graph of the function $f(x) = 5 - (2/x^2)$ is shown in Figure 28. ■

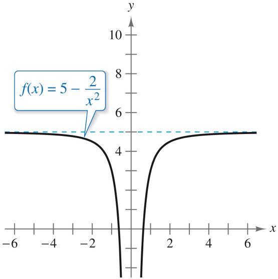


Figure 28: $y = 5$ is a horizontal asymptote.

Example 2 (Finding a limit at infinity)

Find the limit:

$$\lim_{x\to\infty} \frac{2x-1}{x+1}.$$

- Note that both the numerator and the denominator approach infinity as x approaches infinity.

$$\lim_{x\to\infty}\frac{2x-1}{x+1}\begin{array}{l}\Longrightarrow \quad \lim_{x\to\infty}(2x-1)\to\infty \\ \\ \Longrightarrow \quad \lim_{x\to\infty}(x+1)\to\infty\end{array}$$

- This results in $\frac{\infty}{\infty}$ an indeterminate form.
- To resolve this problem, you can divide both the numerator and the denominator by x.
- After dividing, the limit may be evaluated as shown.

$$\lim_{x\to\infty}\frac{2x-1}{x+1}=\lim_{x\to\infty}\frac{\frac{2x-1}{x}}{\frac{x+1}{x}}=\lim_{x\to\infty}\frac{2-\frac{1}{x}}{1+\frac{1}{x}}$$
$$=\frac{\lim_{x\to\infty}2-\lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty}1+\lim_{x\to\infty}\frac{1}{x}}=\frac{2-0}{1+0}=2$$

- So, the line $y=2$ is a horizontal asymptote to the right.

- By taking the limit as $x \to -\infty$, you can see that $y = 2$ is also a horizontal asymptote to the left.
- The graph of the function is shown in Figure 29. ■

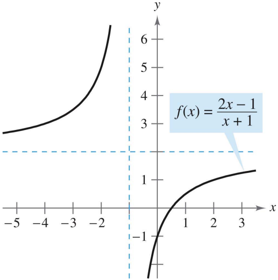


Figure 29: $y = 2$ is a horizontal asymptote of $f(x) = \frac{2x-1}{x+1}$ at $x = \pm\infty$.

Example 3 (A comparison of three rational functions)

Find each limit.

a. $\lim_{x\to\infty} \frac{2x+5}{3x^2+1}$ **b.** $\lim_{x\to\infty} \frac{2x^2+5}{3x^2+1}$ **c.** $\lim_{x\to\infty} \frac{2x^3+5}{3x^2+1}$

- In each case, attempting to evaluate the limit produces the indeterminate form ∞/∞.

a. Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty} \frac{2x+5}{3x^2+1} = \lim_{x\to\infty} \frac{(2/x)+(5/x^2)}{3+(1/x^2)} = \frac{0+0}{3+0} = \frac{0}{3} = 0$$

b. Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty} \frac{2x^2+5}{3x^2+1} = \lim_{x\to\infty} \frac{2+(5/x^2)}{3+(1/x^2)} = \frac{2+0}{3+0} = \frac{2}{3}$$

c. Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty} \frac{2x^3+5}{3x^2+1} = \lim_{x\to\infty} \frac{2x+(5/x^2)}{3+(1/x^2)} = \frac{\infty}{3}$$

- You can conclude that the limit does not exist because the numerator increases without bound while the denominator approaches 3. ■

Guidelines for finding limits at $\pm\infty$ of rational functions

1. If the degree of the numerator is less than the degree of the denominator, then the limit of the rational function is 0.
2. If the degree of the numerator is equal to the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is greater than the degree of the denominator, then the limit of the rational function does not exist.

Example 4 (A function with two horizontal asymptotes)

Find each limit.

a. $\lim_{x\to\infty} \frac{3x-2}{\sqrt{2x^2+1}}$ **b.** $\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}}$

a. For $x > 0$, you can write $x = \sqrt{x^2}$.

So, dividing both the numerator and the denominator by x produces

$$\frac{3x-2}{\sqrt{2x^2+1}} = \frac{\frac{3x-2}{x}}{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}} = \frac{3-\frac{2}{x}}{\sqrt{\frac{2x^2+1}{x^2}}} = \frac{3-\frac{2}{x}}{\sqrt{2+\frac{1}{x^2}}}$$

and you can take the limits as follows.

$$\lim_{x\to\infty} \frac{3x-2}{\sqrt{2x^2+1}} = \lim_{x\to\infty} \frac{3-\frac{2}{x}}{\sqrt{2+\frac{1}{x^2}}} = \frac{3-0}{\sqrt{2+0}} = \frac{3}{\sqrt{2}}$$

b. For $x < 0$, you can write $x = -\sqrt{x^2}$.

So, dividing both the numerator and the denominator by x produces

$$\frac{3x-2}{\sqrt{2x^2+1}} = \frac{\frac{3x-2}{x}}{\frac{\sqrt{2x^2+1}}{-\sqrt{x^2}}} = \frac{3-\frac{2}{x}}{-\sqrt{\frac{2x^2+1}{x^2}}} = \frac{3-\frac{2}{x}}{-\sqrt{2+\frac{1}{x^2}}}$$

and you can take the limits as follows.

$$\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}} = \lim_{x\to-\infty} \frac{3-\frac{2}{x}}{-\sqrt{2+\frac{1}{x^2}}} = \frac{3-0}{-\sqrt{2+0}} = -\frac{3}{\sqrt{2}}$$

The graph of $f(x) = (3x-2)/\sqrt{2x^2+1}$ is shown in Figure 30. ■

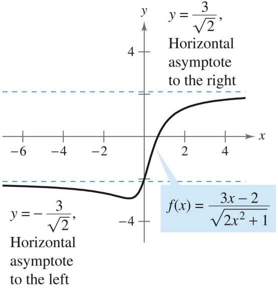


Figure 30: Functions that are not rational may have different right and left horizontal asymptotes.

Example 5 (Limits involving trigonometric functions)

Find each limit.

a. $\lim_{x\to\infty} \sin x$ **b.** $\lim_{x\to\infty} \frac{\sin x}{x}$

a. As x approaches infinity, the sine function oscillates between 1 and -1. So, this limit does not exist.

b. Because $-1 \leq \sin x \leq 1$, it follows that for $x > 0$,

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

where $\lim_{x\to\infty}(-1/x) = 0$ and $\lim_{x\to\infty}(1/x) = 0$. So, by the Squeeze Theorem, you can obtain

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

as shown in Figure 31. ∎

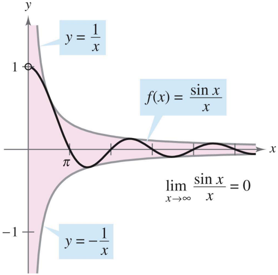


Figure 31: As x increases without bound, $f(x)$ approaches 0.

Example 6 (Oxygen level in a pond)

Suppose that $f(t)$ measures the level of oxygen in a pond, where $f(t) = 1$ is the normal (unpolluted) level and the time t is measured in weeks. When $t = 0$, organic waste is dumped into the pond, and as the waste material oxidizes, the level of oxygen in the pond is

$$f(t) = \frac{t^2 - t + 1}{t^2 + 1}.$$

What percent of the normal level of oxygen exists in the pond after 1 week? After 2 weeks? After 10 weeks? What is the limit as t approaches infinity?

- When $t = 1$, 2, and 10, the levels of oxygen are as shown.

$$f(1) = \frac{1^2 - 1 + 1}{1^2 + 1} = \frac{1}{2} = 50\%$$

$$f(2) = \frac{2^2 - 2 + 1}{2^2 + 1} = \frac{3}{5} = 60\%$$

$$f(10) = \frac{10^2 - 10 + 1}{10^2 + 1} = \frac{91}{101} \approx 90.1\%$$

- To find the limit as t approaches infinity, divide the numerator and the denominator by t^2 to obtain

$$\lim_{t\to\infty} \frac{t^2 - t + 1}{t^2 + 1} = \lim_{t\to\infty} \frac{1 - (1/t) + (1/t^2)}{1 + (1/t^2)} = \frac{1 - 0 + 0}{1 + 0} = 1 = 100\%.$$

■

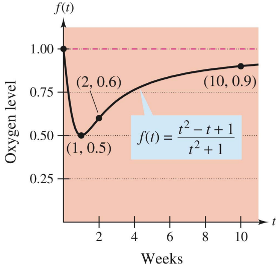


Figure 32: The level of oxygen in a pond approaches the normal level of 1 as t approaches ∞.

Many functions do not approach a finite limit as x increases (or decreases) without bound. For instance, no polynomial function has a finite limit at infinity.

Definition 3.9 (Infinite limits at infinity)

1. The statement $\lim_{x\to\infty} f(x) = \infty$ means that for each positive number M, there is a corresponding number $N > 0$ such that $f(x) > M$ whenever $x > N$.
2. The statement $\lim_{x\to\infty} f(x) = -\infty$ means that for each negative number M, there is a corresponding number $N > 0$ such that $f(x) < M$ whenever $x > N$.

Similar definitions can be given for the statements

$$\lim_{x\to-\infty} f(x) = \infty \text{ and } \lim_{x\to-\infty} f(x) = -\infty.$$

Example 7 (Finding infinite limits at infinity)

Find each limit. **a.** $\lim_{x\to\infty} x^3$ **b.** $\lim_{x\to-\infty} x^3$

a. As x increases without bound, x^3 also increases without bound. So, you can write $\lim_{x\to\infty} x^3 = \infty$.

b. As x decreases without bound, x^3 also increases without bound. So, you can write $\lim_{x\to-\infty} x^3 = -\infty$.

- The graph of $f(x) = x^3$ below illustrates these results. These results agree with the Leading Coefficient Test for polynomial functions. ■

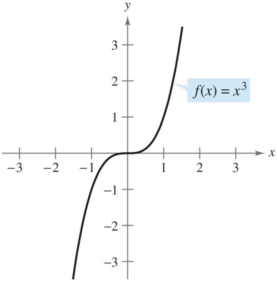


Figure 33: Infinite limits of x^3 at $x = \pm\infty$.

Example 8 (Finding infinite limits at infinity)

Find each limit.

a. $\lim_{x\to\infty} \frac{2x^2-4x}{x+1}$ **b.** $\lim_{x\to-\infty} \frac{2x^2-4x}{x+1}$

a. $\lim_{x\to\infty} \frac{2x^2-4x}{x+1} = \lim_{x\to\infty} \left(2x - 6 + \frac{6}{x+1}\right) = \infty$

b. $\lim_{x\to-\infty} \frac{2x^2-4x}{x+1} = \lim_{x\to-\infty} \left(2x - 6 + \frac{6}{x+1}\right) = -\infty$

- The statements above can be interpreted as saying that as x approaches $\pm\infty$, the function $f(x) = (2x^2 - 4x)/(x+1)$ behaves like the function $g(x) = 2x - 6$.
- In Section 3.6, you will see that this is graphically described by saying that the line $y = 2x - 6$ is a slant asymptote of the graph of f, as shown in Figure 34 ■

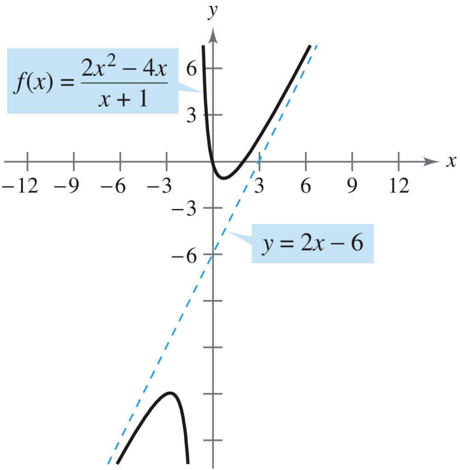


Figure 34: Infinite limits of $\frac{2x^2-4x}{x+1}$ at $x = \pm\infty$.

Table of Contents

Analyzing the graph of a function

So far, you have studied several concepts that are useful in analyzing the graph of a function.

x-intercepts and y-intercepts	(Section P.I)
Symmetry	(Section P.1)
Domain and range	(Section P.1)
Continuity	(Section 1.4)
Vertical asymptotes	(Section .5)
Differentiability	(Section 2.1)
Relative extrema	(Section 1)
Concavity	(Section 3.4)
Points of inflection	(Section 3.4)
Horizontal asymptotes	(Section 3.5)
Infinite limits at infinity	(Section 3.5)

- When you are sketching the graph of a function, either by hand or with a graphing utility, remember that normally you cannot show the entire graph. The decision as to which part of the graph you choose to show is often crucial.
- For instance, which of the viewing windows in Figure 35 better represents the graph of $f(x) = x^3 - 25x^2 + 74x - 20$?

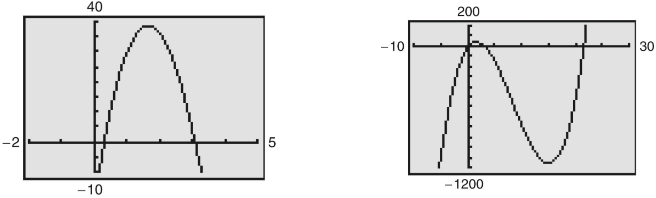


Figure 35: Different views of the graph of $f(x) = x^3 - 25x^2 + 74x - 20$.

- By seeing both views, it is clear that the second viewing window gives a more complete representation of the graph.
- But would a third viewing window reveal other interesting portions of the graph?

- To answer this, you need to use calculus to interpret the first and second derivatives.
- Here are some guidelines for determining a good viewing window for the graph of a function.

Guidelines for analyzing the graph of a function

1. Determine the domain and range of the function.
2. Determine the intercepts, asymptotes, and symmetry of the graph.
3. Locate the x-values for which $f'(x)$ and $f''(x)$ either are zero or do not exist. Use the results to determine relative extrema and points of inflection.

Properties of odd and even functions:

- Odd function: $f(-x) = -f(x)$, $\forall x \in I$, the graph of $f(x)$ symmetry about the origin $(0,0)$, $f'(x)$ is an even function.
- Even function: $f(-x) = f(x)$, $\forall x \in I$, the graph of $f(x)$ symmetry about the y-axis $(x = 0)$, $f'(x)$ is an odd function.

Table 1: Properties of odd and even functions, $f(x) \neq 0$ and $g(x) \neq 0$

(f, g)	$f \pm g$	$f \times g$, $f \div g$
(odd, odd)	odd	even
(even, even)	even	even
(odd, even), (even, odd)	neither	odd

Example 1 (Sketching the graph of a rational function)

Analyze and sketch the graph of $f(x) = \frac{2(x^2-9)}{x^2-4}$.

Domain:	All real numbers except $x = \pm 2$
Range:	$(-\infty, 2) \cup \left[\frac{9}{2}, \infty\right)$
x-intercepts:	$(-3, 0),\ (3, 0)$
y-intercept:	$\left(0, \frac{9}{2}\right)$
Vertical asymptotes:	$x = -2,\ x = 2$
Horizontal asymptotes:	$y = 2$
Symmetry:	With respect to y-axis
First derivative:	$f'(x) = \dfrac{20x}{(x^2-4)^2}$
Second derivative:	$f''(x) = \dfrac{-20(3x^2+4)}{(x^2-4)^3}$

Critical number:	$x = 0$
Possible points of inflection:	None
Test intervals:	$(-\infty, -2),\ (-2, 0),\ (0, 2),\ (2, \infty)$

- The table shows how the test intervals are used to determine several characteristics of the graph.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of graph
$-\infty < x < -2$		$-$	$-$	Decreasing, concave downward
$x = -2$	Undef.	Undef.	Undef.	Vertical asymptote
$-2 < x < 0$		$-$	$+$	Decreasing, concave upward
$x = 0$	$\frac{9}{2}$	0	$+$	Relative minimum
$0 < x < 2$		$+$	$+$	Increasing, concave upward
$x = 2$	Undef.	Undef.	Undef.	Vertical asymptote
$2 < x < \infty$		$+$	$-$	Increasing, concave downward

- The graph of f is shown in Figure 36. ■

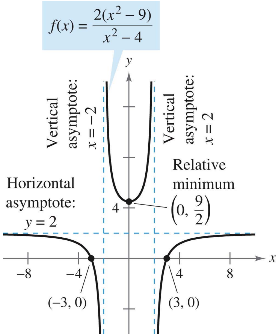


Figure 36: Using calculus, you can be certain that you have determined all characteristics of the graph of $f(x) = \frac{2(x^2-9)}{x^2-4}$.

Example 2 (Sketching the graph of a rational function)

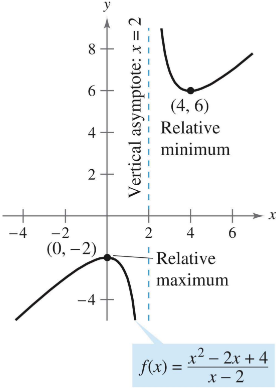
Analyze and sketch the graph of $f(x) = \frac{x^2-2x+4}{x-2}$.

$$\text{Domain:} \quad \text{All real numbers except } x = 2$$

Range:	$(-\infty, -2) \cup [6, \infty)$
x-intercepts:	None
y-intercept:	$(0, -2)$
Vertical asymptote:	$x = 2$
Horizontal asymptotes:	None
Symmetry:	None
End behavior:	$\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = \infty$
First derivative:	$f'(x) = \dfrac{x(x-4)}{(x-2)^2}$
Second derivative:	$f''(x) = \dfrac{8}{(x-2)^3}$
Critical numbers:	$x = 0,\ x = 4$
Possible points of inflection:	None
Test intervals:	$(-\infty, 0),\ (0, 2),\ (2, 4),\ (4, \infty)$

- The analysis of the graph of f is shown below. ■

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of graph
$-\infty < x < 0$		$+$	$-$	Increasing, concave downward
$x = 0$	-2	0	$-$	Relative maximum
$0 < x < 2$		$-$	$-$	Decreasing, concave downward
$x = 2$	Undef.	Undef.	Undef.	Vertical asymptote
$2 < x < 4$		$-$	$+$	Decreasing, concave upward
$x = 4$	6	0	$+$	Relative minimum
$4 < x < \infty$		$+$	$+$	Increasing, concave upward



- The graph of a rational function (having no common factors and whose denominator is of degree 1 or greater) has a slant asymptote if the degree of the numerator exceeds the degree of the denominator by exactly 1.

$$f(x) = \frac{x^2 - 2x + 4}{x - 2} = x + \frac{4}{x - 2}$$

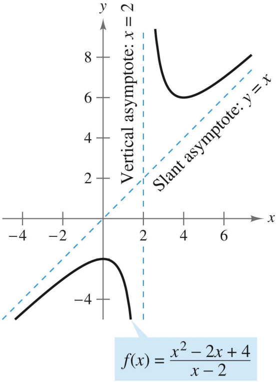


Figure 38: The slant asymptote of $f(x) = \frac{x^2-2x+4}{x-2}$ is $y = x$.

Definition 3.10 (Slant asymptote)

The line $y = mx + b$ is a slant asymptote of the graph of f if

$$\lim_{x\to-\infty} f(x) - (mx+b) = 0 \quad \text{or} \quad \lim_{x\to\infty} f(x) - (mx+b) = 0.$$

Theorem 3.11 (Slant asymptote)

If the line $y = mx + b$ is a slant asymptote of the graph of f, then

$$m = \lim_{x\to-\infty} \frac{f(x)}{x} \qquad b = \lim_{x\to-\infty} f(x) - mx$$

or

$$m = \lim_{x\to\infty} \frac{f(x)}{x} \qquad b = \lim_{x\to\infty} f(x) - mx.$$

Example 3 (Sketching the graph of a radical function)

Analyze and sketch the graph of $f(x) = \frac{x}{\sqrt{x^2+2}}$.

$$f'(x) = \frac{2}{(x^2+2)^{3/2}} \qquad f''(x) = -\frac{6x}{(x^2+2)^{5/2}}$$

- The graph has only one intercept, $(0, 0)$.
- It has no vertical asymptotes, but it has two horizontal asymptotes: $y = 1$ (to the right) and $y = -1$ (to the left).
- The function has no critical numbers and one possible point of inflection (at $x = 0$).
- The domain of the function is all real numbers, and the graph is symmetric with respect to the origin
- The analysis of the graph of f is shown in the table, and the graph of f is shown in Figure 39. ■

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of graph
$-\infty < x < 0$		$+$	$+$	Increasing, concave upward
$x = 0$	0	$+$	0	Point of inflection
$0 < x < \infty$		$+$	$-$	Increasing, concave downward

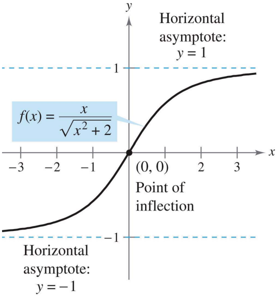


Figure 39: The graph of $f(x) = \frac{x}{\sqrt{x^2+2}}$.

Example 4 (Sketching the graph of a radical function)

Analyze and sketch the graph of $f(x) = 2x^{5/3} - 5x^{4/3}$.

$$f'(x) = \frac{10}{3}x^{1/3}(x^{1/3} - 2) \qquad f''(x) = \frac{20(x^{1/3} - 1)}{9x^{2/3}}$$

- The function has two intercepts: $(0, 0)$ and $\left(\frac{125}{8}, 0\right)$.
- There are no horizontal or vertical asymptotes.
- The function has two critical numbers ($x = 0$ and $x = 8$) and two possible points of inflection ($x = 0$ and $x = 1$).
- The domain is all real numbers. The analysis of the graph of f is shown in the table, and the graph is shown in Figure 40. ■

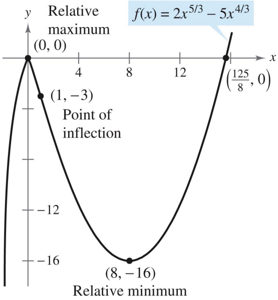


Figure 40: The graph of $f(x) = 2x^{5/3} - 5x^{4/3}$.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of graph
$-\infty < x < 0$		$+$	$-$	Increasing, concave downward
$x = 0$	0	0	Undef.	Relative maximum
$0 < x < 1$		$-$	$-$	Decreasing, concave downward
$x = 1$	-3	$-$	0	Point of inflection
$1 < x < 8$		$-$	$+$	Decreasing, concave upward
$x = 8$	-16	0	$+$	Relative minimum
$8 < x < \infty$		$+$	$+$	Increasing, concave upward

Example 5 (Sketching the graph of a polynomial function)

Analyze and sketch the graph of $f(x) = x^4 - 12x^3 + 48x^2 - 64x$.

- Begin by factoring to obtain

$$f(x) = x^4 - 12x^3 + 48x^2 - 64x = x(x-4)^3.$$

- Then, using the factored form of $f(x)$, you can obtain

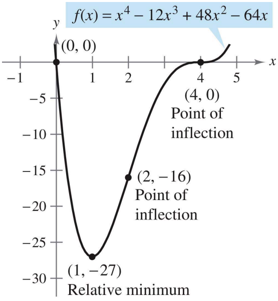
Domain:	All real numbers
Range:	$[-27, \infty)$
x-intercepts:	$(0,0)$, $(4,0)$
y-intercepts:	$(0,0)$
Vertical asymptotes:	None
Horizontal asymptotes:	None
Symmetry:	None
End behavior:	$\lim_{x\to-\infty} f(x) = \infty,\ \lim_{x\to\infty} f(x) = \infty$
First derivative:	$f'(x) = 4(x-1)(x-4)^2$
Second derivative:	$f''(x) = 12(x-4)(x-2)$
Critical numbers:	$x = 1$, $x = 4$

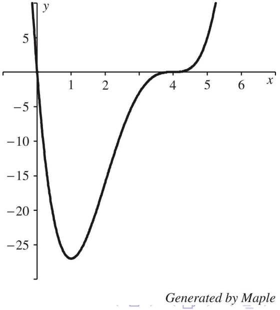
Possible points of inflection: $x = 2, \ x = 4$

Test intervals: $(-\infty, 1), (1, 2), (2, 4), (4, \infty)$

- The analysis of the graph of f is shown below. ■

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of graph
$-\infty < x < 1$		$-$	$+$	Decreasing, concave upward
$x = 1$	-27	0	$+$	Relative minimum
$1 < x < 2$		$+$	$+$	Increasing, concave upward
$x = 2$	-16	$+$	0	Point of inflection
$2 < x < 4$		$+$	$-$	Increasing, concave downward
$x = 4$	0	0	0	Point of inflection
$4 < x < \infty$		$+$	$+$	Increasing, concave upward





The fourth-degree polynomial function in Example 5 has one relative minimum and no relative maxima. In general, a polynomial function of degree n can have at most $n-1$ relative extrema, and at most $n-2$ points of inflection. Moreover, polynomial functions of even degree must have at least one relative extremum.

Example 6 (Sketching the graph of a trigonometric function)

Analyze and sketch the graph of $f(x) = \frac{\cos x}{1+\sin x}$.

- Because the function has a period of 2π, you can restrict the analysis of the graph to any interval of length 2π.
- For convenience, choose $(-\pi/2, 3\pi/2)$

$$\begin{aligned}
\text{Domain:} &\quad \text{All real numbers except } x = \frac{3+4n}{2}\pi \\
\text{Range:} &\quad \text{All real numbers} \\
\text{Period:} &\quad 2\pi \\
x\text{-intercept:} &\quad \left(\frac{\pi}{2}, 0\right) \\
y\text{-intercept:} &\quad (0, 1)
\end{aligned}$$

Vertical asymptotes: $x = -\frac{\pi}{2},\ x = \frac{3\pi}{2}$

Horizontal asymptotes: None

Symmetry: None

First derivative: $f'(x) = \frac{-1}{1 + \sin x}$

Second derivative: $f''(x) = \frac{\cos x}{(1 + \sin x)^2}$

Critical numbers: None

Possible points of inflection: $x = \frac{\pi}{2}$

Test intervals: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right),\ \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

- The analysis of the graph off on the interval $(-\pi/2, 3\pi/2)$ is shown in the table, and the graph is shown in Figure 42.
- Compare this with the graph generated by the computer algebra system in Figure 42.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of graph
$x = -\frac{\pi}{2}$	Undef.	Undef.	Undef.	Vertical asymptote
$-\frac{\pi}{2} < x < \frac{\pi}{2}$		$-$	$+$	Decreasing, concave upward
$x = \frac{\pi}{2}$	0	$-$	0	Point of inflection
$\frac{\pi}{2} < x < \frac{3\pi}{2}$		$-$	$-$	Decreasing, concave downward
$x = \frac{3\pi}{2}$	Undef.	Undef.	Undef.	Vertical asymptote

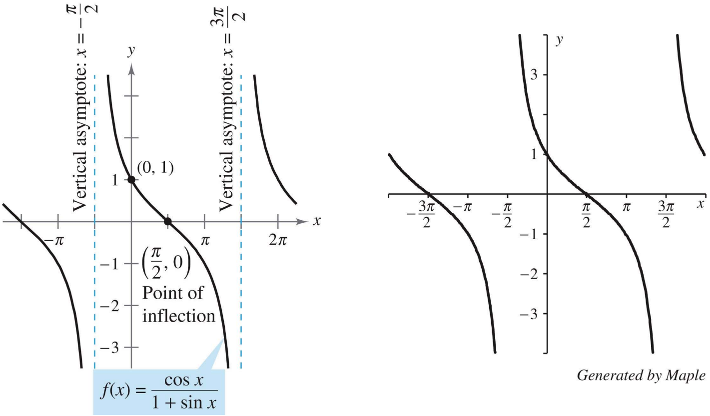


Figure 42: The graph of $f(x) = \frac{\cos x}{1+\sin x}$.

Table of Contents

Applied minimum and maximum problems

- One of the most common applications of calculus involves the determination of minimum and maximum values.

Example 1 (Finding maximum volume)

A manufacturer wants to design an open box having a square base and a surface area of 108 square centimeters, as shown in Figure 43. What dimensions will produce a box with maximum volume?

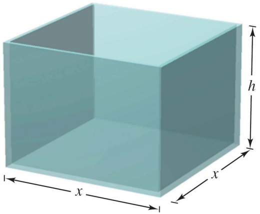


Figure 43: Open box with square base: Surface area $S = x^2 + 4xh = 108$.

- Because the box has a square base, its volume is

$$V = x^2 h. \qquad \text{Primary equation}$$

- This equation is called the primary equation because it gives a formula for the quantity to be optimized.
- The surface area of the box is

$$S = (\text{area of base}) + (\text{area of four sides}) \quad S = x^2 + 4xh = 108.$$

Secondary equation

- Because V is to be maximized, you want to write V as a function of just one variable. To do this, you can solve the equation $x^2 + 4xh = 108$ for h in terms of x to obtain $h = (108 - x^2)/(4x)$.
- Substituting into the primary equation produces

$$V = x^2 h = x^2 \left(\frac{108 - x^2}{4x} \right) = 27x - \frac{x^3}{4}.$$

- Before finding which x-value will yield a maximum value of V, you should determine the feasible domain.

- That is, what values of x make sense in this problem?
- You know that $V \geq 0$. You also know that x must be nonnegative and that the area of the base ($A = x^2$) is at most 108.
- So, the feasible domain is

$$0 \leq x \leq \sqrt{108}. \qquad \text{Feasible domain}$$

- To maximize V, find the critical numbers of the volume function on the interval $(0, \sqrt{108})$.

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 27 - \frac{3x^2}{4} = 0 \qquad 3x^2 = 108 \qquad x = \pm 6$$

- So, the critical numbers are $x = \pm 6$.
- You do not need to consider $x = -6$ because it is outside the domain.
- Evaluating V at the critical number $x = 6$ and at the endpoints of the domain produces $V(0) = 0$, $V(6) = 108$, and $V(\sqrt{108}) = 0$.
- So, V is maximum when $x = 6$ and the dimensions of the box are $6 \times 6 \times 3$ centimeters.

Guidelines for solving applied minimum and maximum problems

1. Identify all given quantities and all quantities to be determined If possible, make a sketch.
2. Write a primary equation for the quantity that is to be maximized or minimized. (A review of several useful formulas from geometry is presented inside the back cover.)
3. Reduce the primary equation to one having a single independent variable. This may involve the use of secondary equations relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by the calculus techniques discussed in Sections 3.1 through 3.4.

Example 2 (Finding minimum distance)

Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?

- Figure 44 shows that there are two points at a minimum distance from the point $(0, 2)$.
- The distance between the point $(0, 2)$ and a point (x, y) on the graph of $y = 4 - x^2$ is given by

$$d = \sqrt{(x-0)^2 + (y-2)^2}. \qquad \text{Primary equation}$$

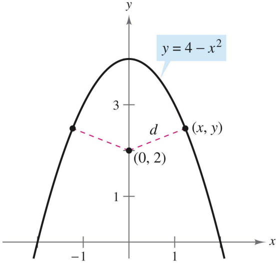


Figure 44: The quantity to be minimized is distance $d = \sqrt{(x-0)^2 + (y-2)^2}$.

- Using the secondary equation $y = 4 - x^2$, you can rewrite the primary equation as

$$d = \sqrt{x^2 + (4 - x^2 - 2)^2} = \sqrt{x^4 - 3x^2 + 4}.$$

- Because d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of $f(x) = x^4 - 3x^2 + 4$.
- Note that the domain of f is the entire real line.
- So, there are no endpoints of the domain to consider.
- Moreover, setting $f'(x)$ equal to 0 yields

$$f'(x) = 4x^3 - 6x = 2x(2x^2 - 3) = 0 \qquad x = 0,\ \sqrt{\frac{3}{2}},\ -\sqrt{\frac{3}{2}}.$$

- The First Derivative Test verifies that $x = 0$ yields a relative maximum, whereas both $x = \sqrt{3/2}$ and $x = -\sqrt{3/2}$ yield a minimum distance.
- So, the closest points are $(\sqrt{3/2}, 5/2)$ and $(-\sqrt{3/2}, 5/2)$. ∎

Example 5 (An endpoint maximum)

Four meters of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?

- The total area (see Figure 45) is given by

$$A = (\text{area of square}) + (\text{area of circle}) \qquad A = x^2 + \pi r^2.$$

Primary equation

- Because the total length of wire is 4 meters, you obtain

$$4 = (\text{perimeter of square}) + (\text{circumference of circle}) \quad 4 = 4x + 2\pi r.$$

Secondary equation

- So, $r = 2(1-x)/\pi$, and by substituting into the primary equation you have

$$A = x^2 + \pi\left[\frac{2(1-x)}{\pi}\right]^2 = x^2 + \frac{4(1-x)^2}{\pi} = \frac{1}{\pi}\left[(\pi+4)x^2 - 8x + 4\right].$$

- The feasible domain is $0 \le x \le 1$ restricted by the square's perimeter.

- Because
$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{2(\pi+4)x-8}{\pi}$$
the only critical number in $(0, 1)$ is $x = 4/(\pi+4) \approx 0.56$.
- So, using
$$A(0) \approx 1.27, \quad A(0.56) \approx 0.56, \quad \text{and} \quad A(1) = 1$$
you can conclude that the maximum area occurs when $x = 0$.
- That is, all the wire is used for the circle. ■

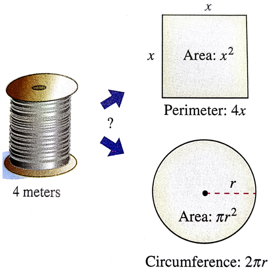


Figure 45: The quantity to be maximized is area: $A = x^2 + \pi r^2$.