Chapter 2 Differentiation

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The tangent line problem

- Calculus grew out of four major problems that European mathematicians were working on during the seventeenth century.
  1. The tangent line problem
  2. The velocity and acceleration problem
  3. The minimum and maximum problem
  4. The area problem
- Each problem involves the notion of a limit, and calculus can be introduced with any of the four problems. Essentially, the problem of finding the tangent line at point P boils down to the problem of finding the slope of the tangent line at point P.
- You can approximate this slope using a secant line through the point of tangency and a second point on the curve, as shown in Figure 1.

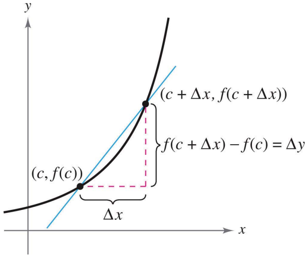

Figure 1: The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$.

- If $(c, f(c))$ is the point of tangency and $(c + \Delta x, f(c + \Delta x))$ is a second point on the graph of f,then the slope of the secant line through the two points is given by substitution into the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c}$$

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

- The right-hand side of this equation is a difference quotient
- The denominator Δx is the change in x, and the numerator $\Delta y = f(c + \Delta x) - f(c)$ is the change in y.

Definition 2.1 (Tangent line with slope m)

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the tangent line to the graph of f at the point $(c, f(c))$.

- The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the slope of the graph of f at $x = c$.

Example 1 (The slope of the graph of a linear function)

Find the slope of the graph of $f(x) = 2x - 3$ at the point $(2, 1)$.

- To find the slope of the graph of f when $c = 2$, you can apply the definition of the slope of a tangent line, as shown.

$$\begin{aligned}\lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{[2(2+\Delta x) - 3] - [2(2) - 3]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{4 + 2\Delta x - 3 - 4 + 3}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 2 = 2\end{aligned}$$

- The slope of f at $(c, f(c)) = (2, 1)$ is $m = 2$, as shown in Figure 2. ■

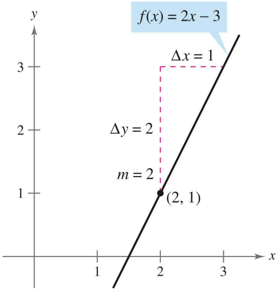

Figure 2: The slope of $f(x) = 2x - 3$ at $(2, 1)$ is $m = 2$.

Example 2 (Tangent lines to the graph of a nonlinear function)

Find the slopes of the tangent lines to the graph of $f(x) = x^2 + 1$ at the points $(0, 1)$ and $(-1, 2)$, as shown in Figure 3.

- Let $(c, f(c))$ represent an arbitrary point on the graph of f. Then the slope of the tangent line at $(c, f(c))$ is given by

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[(c+\Delta x)^2 + 1\right] - \left[c^2 + 1\right]}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{c^2 + 2c(\Delta x) + (\Delta x)^2 + 1 - c^2 - 1}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (2c + \Delta x) = 2c.$$

So, the slope at any point $(c, f(c))$ on the graph of f is $m = 2c$. At the point $(0, 1)$, the slope is $m = 2(0) = 0$, and at $(-1, 2)$, the slope is $m = 2(-1) = -2$. ■

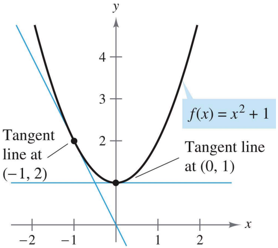

Figure 3: The slope of $f(x) = x^2 + 1$ at any point $(c, f(c))$ is $m = 2c$.

- The definition of a tangent line to a curve does not cover the possibility of a vertical tangent line. For vertical tangent lines, you can use the following definition. If f is continuous at c and

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = -\infty$$

  the vertical line $x = c$ passing through $(c, f(c))$ is a vertical tangent line to the graph of f.
- For example, the function shown in Figure 4 has a vertical tangent line at $(c, f(c))$. If the domain of f is the closed interval $[a, b]$, you can extend the definition of a vertical tangent line to include the endpoints by considering continuity and limits from the right (for $x = a$) and from the left (for $x = b$).

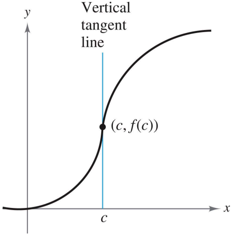

Figure 4: The graph of f has a vertical tangent line at $(c, f(c))$.

- The above limit is also used to define the differentiation.

Definition 2.2 (The derivative of a function)

The derivative of f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

- Be sure you see that the derivative of a function of x is also a function of x. This "new" function gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, provided that the graph has a tangent line at this point. The process of finding the derivative of a function is called differentiation.
- A function is differentiable at x if its derivative exists at x and is differentiable on an open interval (a, b) if it is differentiable at every point in the interval.
- In addition to $f'(x)$, which is read as "f prime of x," other notations are used to denote the derivative of $y = f(x)$. The most common are

$$f'(x), \quad \frac{\mathrm{d}y}{\mathrm{d}x}, \quad y', \quad \frac{\mathrm{d}}{\mathrm{d}x}[f(x)], \quad D_x[y]. \quad \text{Notation for derivatives}$$

- The notation $\mathrm{d}y/\mathrm{d}x$ is read as "the derivative of y with respect to x" or simply "$\mathrm{d}y$, $\mathrm{d}x$." Using limit notation, you can write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x).$$

Example 3 (Finding the derivative by the limit process)

Find the derivative of $f(x) = x^3 + 2x$.

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 + 2(x+\Delta x) - (x^3 + 2x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x + 2\Delta x - x^3 - 2x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x\left[3x^2 + 3x\Delta x + (\Delta x)^2 + 2\right]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[3x^2 + 3x\Delta x + (\Delta x)^2 + 2\right] = 3x^2 + 2
\end{aligned}
$$

Note that the derivative of function f is a function, which can be used to find the slope of tangent line at the point $(x, f(x))$ on the graph of f. ■

Example 4 (Using the derivative to find the slope at a point)

Find $f'(x)$ for $f(x) = \sqrt{x}$. Then find the slopes of the graph of f at the points $(1,1)$ and $(4,2)$. Discuss the behavior of f at $(0,0)$.

- Use the procedure for rationalizing numerators

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(\frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \right) \left(\frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}} \right) \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x) - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} \\
&= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}, \quad x > 0
\end{aligned}$$

- At the point $(1,1)$, the slope is $f'(1) = 1/2$. At the point $(4,2)$, the slope is $f'(4) = 1/4$. See Figure 5.
- At the point $(0,0)$, the slope is undefined. Moreover, the graph of f has a vertical tangent line at $(0,0)$. ■

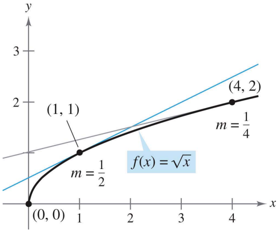

Figure 5: The slope of $f(x) = \sqrt{x}$ at $(x, f(x))$, $x > 0$, is $m = 1/(2\sqrt{x})$.

Example 5 (Finding the derivative of a function)

Find the derivative with respect to t for the function $y = 2/t$.

Let $y = f(t)$, you obtain

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\frac{2}{t+\Delta t} - \frac{2}{t}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\frac{2t-2(t+\Delta t)}{t(t+\Delta t)}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{-2\Delta t}{\Delta t(t)(t+\Delta t)} \\
&= \lim_{\Delta t \to 0} \frac{-2}{t(t+\Delta t)} = -\frac{2}{t^2}.
\end{aligned}$$

■

Differentiability and continuity

- The following alternative limit form of the derivative is useful in investigating the relationship between differentiability and continuity. The derivative of f at c is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \text{Alternative form of derivative}$$

provided this limit exists (see Figure 6).

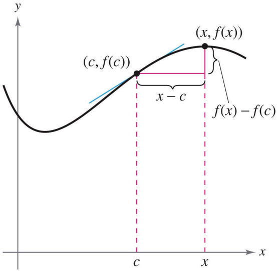

Figure 6: As x approaches c, the secant line approaches the tangent line.

The derivative of f at c is given by

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

Let $x = c + \Delta x$. Then $x \to c$ as $\Delta x \to 0$. So, replacing $c + \Delta x$ by x, you have

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

□

- Note that the existence of the limit in this alternative form requires that the one-sided limits

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

exist and are equal. These one-sided limits are called the derivatives from the left and derivatives from the right, respectively.

- It follows that f is differentiable on the closed interval $[a, b]$ if it is differentiable on (a, b) and if the derivative from the right at a and the derivative from the left at b both exist.

- If a function is not continuous at $x = c$, it is also not differentiable at $x = c$. For instance, the greatest integer function $f(x) = \lfloor x \rfloor$ is not continuous at $x = 0$, and so it is not differentiable at $x = 0$

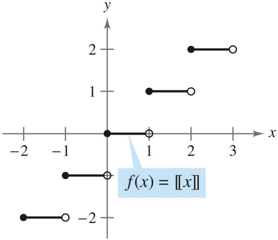

Figure 7: The greatest integer function is not differentiable at $x = 0$, because it is not continuous at $x = 0$.

- You can verify this by observing that

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{\lfloor x \rfloor - 0}{x}$$

$$= \lim_{x \to 0^-} \frac{-1 - 0}{x} = \infty \quad \text{Derivative from the left}$$

and

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\lfloor x \rfloor - 0}{x}$$
$$= \lim_{x\to 0^+} \frac{0-0}{x} = 0. \quad \text{Derivative from the right}$$

Although it is true that differentiability implies continuity the converse is not true.

Example 6 (A graph with a sharp turn)

- The function $f(x) = |x-2|$ shown in Figure 8 is continuous at $x = 2$.
- However, the one-sided limits

$$\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{|x-2|-0}{x-2} = -1 \quad \text{Derivative from the left}$$

$$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{|x-2|-0}{x-2} = 1 \quad \text{Derivative from the right}$$

are not equal.

- So, f is not differentiable at $x = 2$ and the graph of f does not have a tangent line at the point $(2, 0)$. ■

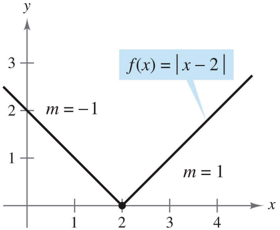

Figure 8: $f(x) = |x - 2|$ is not differentiable at $x = 2$, because the derivatives from the left and from the right are not equal.

Example 7 (A graph with a vertical tangent line)

- The function $f(x) = x^{1/3}$ is continuous at $x = 0$, as shown in Figure 9. However, because the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^{1/3} - 0}{x} = \lim_{x\to 0} \frac{1}{x^{2/3}} = \infty$$

is infinite, you can conclude that the tangent line is vertical at $x = 0$. So, f is not differentiable at $x = 0$. ■

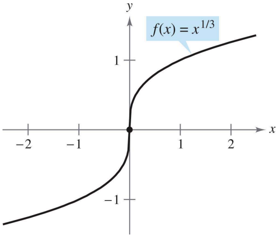

Figure 9: The graph of f has a vertical tangent line at $(c, f(c))$.

Theorem 2.1 (Differentiability implies continuity)

If f is differentiable at $x = c$, then f is continuous at $x = c$.

- You can prove that f continuous at $x = c$ by showing that $f(x)$ approaches $f(c)$ as $x \to c$.
- To do this, use the differentiability of f at $x = c$ and consider the following limit.

$$\lim_{x\to c}[f(x) - f(c)] = \lim_{x\to c}\left[(x-c)\left(\frac{f(x)-f(c)}{x-c}\right)\right]$$
$$= \left[\lim_{x\to c}(x-c)\right]\left[\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right]$$
$$= (0)[f'(c)] = 0$$

Because the difference $f(x) - f(c)$ approaches zero as $x \to c$, you can conclude that $\lim_{x\to c} f(x) = f(c)$.So, f is continuous at $x = c$. □

The following statements summarize the relationship between continuity and differentiability.

1. If a function is differentiable at $x = c$, then it is continuous at $x = c$. So, differentiability implies continuity.
2. It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, continuity does not imply differentiability. eg. $f(x) = |x^2 - 1|$ at $x = \pm 1$.

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The Constant Rule

Theorem 2.2 (The Constant Rule)

The derivative of a constant function is 0. *That is, if c is a real number, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}[c] = 0.$$

Let $f(x) = c$. Then, by the limit definition of the derivative,

$$\frac{\mathrm{d}}{\mathrm{d}x}[c] = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0.$$

□

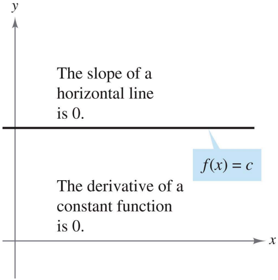

Figure 10: Notice that the Constant Rule is equivalent to saying that the slope of a horizontal line is 0.

Example 1 (Using the Constant Rule)

Function	Derivative
a. $y = 7$	$\frac{dy}{dx} = 0$
b. $f(x) = 0$	$f'(x) = 0$
c. $s(t) = -3$	$s'(t) = 0$
d. $y = k\pi^2$, k is constant	$y' = 0$ ■

The Power Rule

- Before proving the next rule, it is important to review the procedure for expanding a binomial.

$$(x+\Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$
$$(x+\Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

- The general binomial expansion for a positive integer n is

$$(x+\Delta x)^n = x^n + nx^{n-1}(\Delta x) + \underbrace{\frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n}_{(\Delta x)^2 \text{ is a factor of these terms.}}.$$

- This binomial expansion is used in proving a special case of the Power Rule.

Theorem 2.3 (The Power Rule)

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0.

- We will proves the case for which n is a positive integer greater than 1.
- You will prove the case for $n = 1$. In Section 2.3, it proves the case for which n is a negative integer. In Section 2.5, you are asked to prove the case for which n is rational. In Section 5.5, the Power Rule will be extended to cover irrational values of n.

- If n is a positive integer greater than 1, then the binomial expansion produces

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}[x^n] \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}(\Delta x) + \frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[nx^{n-1} + \frac{n(n-1)x^{n-2}}{2}(\Delta x) + \cdots + (\Delta x)^{n-1} \right] \\
&= nx^{n-1} + 0 + 0 + \cdots + 0 = nx^{n-1}.
\end{aligned}$$

□

- When using the Power Rule, the case for which $n = 1$ is best thought of as a separate differentiation rule. That is,

$$\frac{\mathrm{d}}{\mathrm{d}x}[x] = 1. \qquad \text{Power Rule when } n = 1$$

- This rule is consistent with the fact that the slope of the line $y = x$ is 1, as shown in Figure 11.

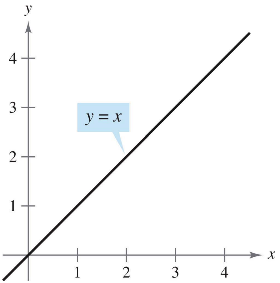

Figure 11: The slope of the line $y = x$ is 1.

Example 2 (Using the Power Rule)

	Function	Derivative
a.	$f(x) = x^3$	$f'(x) = 3x^2$
b.	$g(x) = \sqrt[3]{x}$	$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[x^{1/3}\right] = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$
c.	$y = \frac{1}{x^2}$	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-2}\right] = (-2)x^{-3} = -\frac{2}{x^3}$

■

In Example 2(c), note that before differentiating, $\frac{1}{x^2}$ was rewritten as x^{-2}. Rewriting is the first step in many differentiation problems.

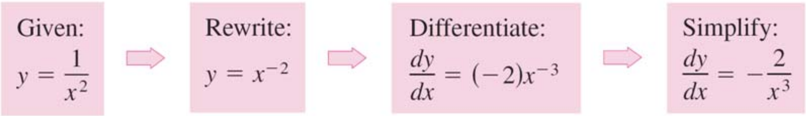

Figure 12: Steps for Power Rule.

Example 3 (Finding the slope of a graph)

Find the slope of the graph of $f(x) = x^4$ when
a. $x = -1$ **b.** $x = 0$ **c.** $x = 1$.

- The slope of a graph at a point is the value of the derivative at that point. The derivative of f is $f'(x) = 4x^3$.
  a. $f'(-1) = -4$ **b.** $f'(0) = 0$ **c.** $f'(1) = 4$. ■

Example 4 (Finding an equation of a tangent line)

Find an equation of the tangent line to the graph of $f(x) = x^2$ when $x = -2$.

- To find the point on the graph of f, evaluate the original function at $x = -2$.

$$(-2, f(-2)) = (-2, 4) \qquad \text{Point on graph}$$

- To find the slope of the graph when $x = -2$, evaluate the derivative, $f'(x) = 2x$, at $x = -2$.

$$m = f'(-2) = -4 \qquad \text{Slope of graph at } (-2, 4)$$

- Now, using the point-slope form of the equation of a line, you can write

$$y - y_1 = m(x - x_1) \quad y - 4 = -4(x - (-2)) \quad y = -4x - 4. \quad \blacksquare$$

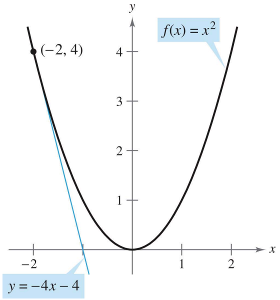

Figure 13: The line $y = -4x - 4$ is tangent to the graph of $f(x) = x^2$ at the point $(-2, 4)$.

The Constant Multiple Rule

Theorem 2.4 (The Constant Multiple Rule)

If f is a differentiable function and c is a real number, then cf is also differentiable and $\frac{\mathrm{d}}{\mathrm{d}x}[cf(x)] = cf'(x)$.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[cf(x)] &= \lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x} = \lim_{\Delta x \to 0} c \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] \\
&= c \left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \right] = cf'(x)
\end{aligned}
$$

□

Example 5 (Using the Constant Multiple Rule)

Function	Derivative
a. $y = 5x^3$	$\frac{dy}{dx} = \frac{d}{dx}[5x^3] = 5\frac{d}{dx}[x^3] = 5(3)x^2 = 15x^2$
b. $y = \frac{2}{x}$	$\frac{dy}{dx} = \frac{d}{dx}[2x^{-1}] = 2\frac{d}{dx}[x^{-1}] = 2(-1)x^{-2} = -\frac{2}{x^2}$
c. $f(t) = \frac{4t^2}{5}$	$f'(t) = \frac{d}{dt}\left[\frac{4}{5}t^2\right] = \frac{4}{5}\frac{d}{dt}\left[t^2\right] = \frac{4}{5}(2t) = \frac{8}{5}t$
d. $y = 2\sqrt{x}$	$\frac{dy}{dx} = \frac{d}{dx}\left[2x^{1/2}\right] = 2\left(\frac{1}{2}x^{-1/2}\right) = x^{-1/2} = \frac{1}{\sqrt{x}}$
e. $y = \frac{1}{2\sqrt[3]{x^2}}$	$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{2}x^{-2/3}\right] = \frac{1}{2}\left(-\frac{2}{3}\right)x^{-5/3} = -\frac{1}{3x^{5/3}}$
f. $y = -\frac{3x}{2}$	$y' = \frac{d}{dx}\left[-\frac{3}{2}x\right] = -\frac{3}{2}(1) = -\frac{3}{2}$

■

The Constant Multiple Rule and the Power Rule can be combined into one rule. The combination rule is

$$\frac{\mathrm{d}}{\mathrm{d}x}[cx^n] = cnx^{n-1}.$$

Example 6 (Using parentheses when differentiating)

	Function	Rewrite	Differentiate	Simplify
a.	$y = \frac{5}{2x^3}$	$y = \frac{5}{2}x^{-3}$	$y' = \frac{5}{2}(-3)x^{-4}$	$y' = -\frac{15}{2x^4}$
b.	$y = \frac{5}{(2x)^3}$	$y = \frac{5}{8}x^{-3}$	$y' = \frac{5}{8}(-3)x^{-4}$	$y' = -\frac{15}{8x^4}$
c.	$y = \frac{7}{3x^{-2}}$	$y = \frac{7}{3}x^{2}$	$y' = \frac{7}{3}(2x)$	$y' = \frac{14x}{3}$
d.	$y = \frac{7}{(3x)^{-2}}$	$y = 63(x^2)$	$y' = 63(2x)$	$y' = 126x$

■

Theorem 2.5 (The Sum and Difference Rules)

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f+g$ (or $f-g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)+g(x)] = f'(x)+g'(x) \qquad \textit{Sum Rule}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)-g(x)] = f'(x)-g'(x) \qquad \textit{Difference Rule}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[f(x)+g(x)] &= \lim_{\Delta x\to 0}\frac{[f(x+\Delta x)+g(x+\Delta x)]-[f(x)+g(x)]}{\Delta x}\\
&= \lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}+\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}\\
&= f'(x)+g'(x)
\end{aligned}$$

A proof of the difference Rule can be proved in a similar way. □

Example 7 (Using the Sum and Difference Rules)

	Function	Derivative
a.	$f(x) = x^3 - 4x + 5$	$f'(x) = 3x^2 - 4$
b.	$g(x) = -\frac{x^4}{2} + 3x^3 - 2x$	$g'(x) = -2x^3 + 9x^2 - 2$
c.	$y = \frac{3x^2 - x + 1}{x} = 3x - 1 + \frac{1}{x}$	$y' = 3 - \frac{1}{x^2} = \frac{3x^2 - 1}{x^2}$

■

Theorem 2.6 (Derivatives of the sine and cosine functions)

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\cos x] = -\sin x$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] &= \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cos x \sin \Delta x - (\sin x)(1 - \cos \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[(\cos x)\left(\frac{\sin \Delta x}{\Delta x}\right) - (\sin x)\left(\frac{1 - \cos \Delta x}{\Delta x}\right) \right] \\
&= \cos x \left(\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \right) - \sin x \left(\lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} \right) \\
&= (\cos x)(1) - (\sin x)(0) = \cos x
\end{aligned}$$

- This differentiation rule is shown graphically in Figure 14.
- Note that for each x, the slope of the sine curve is equal to the value of the cosine.
- The proof of the second rule is left as an exercise (see Exercise 114). □

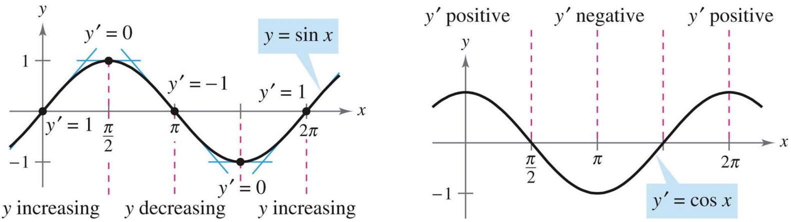

Figure 14: The derivative of the sine function is the cosine function.

Example 8 (Derivatives involving sines and cosines)

Function	Derivative
a. $y = 2\sin x$	$y' = 2\cos x$
b. $y = \frac{\sin x}{2} = \frac{1}{2}\sin x$	$y' = \frac{1}{2}\cos x = \frac{\cos x}{2}$
c. $y = x + \cos x$	$y' = 1 - \sin x$
d. $y = \cos x - \frac{\pi}{3}\sin x$	$y' = -\sin x - \frac{\pi}{3}\cos x$

■

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The Product Rule

Theorem 2.7 (The Product Rule)

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

- Some mathematical proofs, such as the proof of the Sum Rule, are straightforward. Others involve clever steps that may appear unmotivated to a reader. This proof involves such a step-subtracting and adding the same quantity.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x)}{\Delta x} \\
&\quad + \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[f(x+\Delta x)\frac{g(x+\Delta x) - g(x)}{\Delta x} + g(x)\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \left[f(x+\Delta x)\frac{g(x+\Delta x) - g(x)}{\Delta x} \right] \\
&\quad + \lim_{\Delta x \to 0} \left[g(x)\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} f(x+\Delta x) \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&\quad + \lim_{\Delta x \to 0} g(x) \cdot \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}
$$

- If f, g, and h are differentiable functions of x, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

Example 1 (Using the Product Rule)

Find the derivative of $h(x) = (3x - 2x^2)(5 + 4x)$.

$$\begin{aligned} h'(x) &= \underbrace{(3x - 2x^2)}_{\text{First}} \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}[5 + 4x]}_{\text{Derivative of second}} + \underbrace{(5 + 4x)}_{\text{Second}} \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}[3x - 2x^2]}_{\text{Derivative of first}} \\ &= (3x - 2x^2)(4) + (5 + 4x)(3 - 4x) \\ &= (12x - 8x^2) + (15 - 8x - 16x^2) = -24x^2 + 4x + 15 \end{aligned}$$

■

Example 2 (Using the Product Rule)

Find the derivative of
$y = 3x^2 \sin x$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[3x^2 \sin x\right] = 3x^2 \frac{\mathrm{d}}{\mathrm{d}x}\left[\sin x\right] + \sin x \frac{\mathrm{d}}{\mathrm{d}x}\left[3x^2\right]$$

$$= 3x^2 \cos x + (\sin x)(6x) = 3x(x \cos x + 2 \sin x) \qquad \blacksquare$$

Example 3 (Using the Product Rule)

Find the derivative of $y = 2x \cos x - 2 \sin x$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \underbrace{2x\left(\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos x\right]\right) + (\cos x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\left[2x\right]\right)}_{\text{Product Rule}} - \underbrace{2\left(\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin x\right]\right)}_{\text{Constant Multiple Rule}}$$

$$= 2x(-\sin x) + (\cos x)(2) - 2(\cos x) = -2x \sin x \qquad \blacksquare$$

The Quotient Rule

Theorem 2.8 (The Quotient Rule)

The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

- As with the proof of Theorem 2.7, the key to this proof is subtracting and adding the same quantity.

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] \\
&= \lim_{\Delta x \to 0} \frac{\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x+\Delta x)}{\Delta x g(x)g(x+\Delta x)} \\
&= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x) + f(x)g(x) - f(x)g(x+\Delta x)}{\Delta x g(x)g(x+\Delta x)} \\
&= \lim_{\Delta x \to 0} \frac{g(x)[f(x+\Delta x) - f(x)]}{\Delta x g(x)g(x+\Delta x)} - \lim_{\Delta x \to 0} \frac{f(x)[g(x+\Delta x) - g(x)]}{\Delta x g(x)g(x+\Delta x)} \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}
\end{aligned}
$$

Note that $\lim_{\Delta x \to 0} g(x+\Delta x) = g(x)$ because g is given to be differentiable and therefore is continuous. □

Example 4 (Using the Quotient Rule)

Find the derivative of $y = \frac{5x-2}{x^2+1}$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{5x-2}{x^2+1}\right] = \frac{(x^2+1)\frac{\mathrm{d}}{\mathrm{d}x}\,[5x-2] - (5x-2)\frac{\mathrm{d}}{\mathrm{d}x}\,[x^2+1]}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(5) - (5x-2)(2x)}{(x^2+1)^2} = \frac{(5x^2+5) - (10x^2-4x)}{(x^2+1)^2}$$

$$= \frac{-5x^2+4x+5}{(x^2+1)^2} \qquad \blacksquare$$

Example 5 (Rewriting before differentiating)

Find an equation of the tangent line to the graph of $f(x) = \frac{3-(1/x)}{x+5}$ at $(-1, 1)$.

$$f(x) = \frac{3-(1/x)}{x+5} = \frac{3x-1}{x^2+5x}$$

$$f'(x) = \frac{(x^2+5x)(3)-(3x-1)(2x+5)}{(x^2+5x)^2}$$

$$= \frac{-3x^2+2x+5}{(x^2+5x)^2}$$

To find the slope at $(-1, 1)$, evaluate $f'(-1) = 0$. The equation of the tangent line at $(-1, 1)$ is $y = 1$. ■

Example 6 (Using the Constant Multiple Rule)

Function	Rewrite	Differentiate	Simplify
a. $y = \frac{x^2+3x}{6}$	$y = \frac{1}{6}(x^2 + 3x)$	$y' = \frac{1}{6}(2x + 3)$	$y' = \frac{2x+3}{6}$
b. $y = \frac{5x^4}{8}$	$y = \frac{5}{8}(x^4)$	$y' = \frac{5}{8}(4x^3)$	$y' = \frac{5}{2}x^3$
c. $y = \frac{-3(3x-2x^2)}{7x}$	$y = -\frac{3}{7}(3 - 2x)$	$y' = -\frac{3}{7}(-2)$	$y' = \frac{6}{7}$
d. $y = \frac{9}{5x^2}$	$y = \frac{9}{5}(x^{-2})$	$y' = \frac{9}{5}(-2x^{-3})$	$y' = -\frac{18}{5x^3}$

■

Example 7 (Proof of the Power Rule (negative integer exponents))

If n is a negative integer, there exists a positive integer k such that $n = -k$.

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x^k}\right] = \frac{x^k(0) - (1)(kx^{k-1})}{x^{2k}} = \frac{0 - kx^{k-1}}{x^{2k}}$$
$$= -kx^{-k-1} = nx^{n-1}$$

□

Derivatives of trigonometric functions

- Knowing the derivatives of the sine and cosine functions, you can use the Quotient Rule to find the derivatives of the four remaining trigonometric functions.

Theorem 2.9 (Derivatives of trigonometric functions)

$\frac{\mathrm{d}}{\mathrm{d}x}[\tan x] = \sec^2 x$ $\qquad$ $\frac{\mathrm{d}}{\mathrm{d}x}[\cot x] = -\csc^2 x$

$\frac{\mathrm{d}}{\mathrm{d}x}[\sec x] = \sec x \tan x$ $\qquad$ $\frac{\mathrm{d}}{\mathrm{d}x}[\csc x] = -\csc x \cot x$

Considering $\tan x = (\sin x)/(\cos x)$ and applying the Quotient Rule, you obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan x] = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

The proofs of the other three parts of the theorem are left as an exercise (see Exercise 87).

Example 8 (Differentiating trigonometric functions)

a. $y = x - \tan x \quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 - \sec^2 x$

b. $y = x \sec x \qquad y' = x(\sec x \tan x) + (\sec x)(1) = (\sec x)(1 + x \tan x)$

Example 9 (Different forms of a derivative)

Differentiate both forms of $y = \frac{1-\cos x}{\sin x} = \csc x - \cot x$.

$$
\begin{aligned}
\text{First form } y &= \frac{1-\cos x}{\sin x} \\
y' &= \frac{(\sin x)(\sin x) - (1-\cos x)(\cos x)}{\sin^2 x} \\
&= \frac{\sin^2 x + \cos^2 x - \cos x}{\sin^2 x} = \frac{1-\cos x}{\sin^2 x} \\
&= \frac{1}{\sin^2 x} - \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) = \csc^2 x - \csc x \cot x \\
\text{Second form } y &= \csc x - \cot x \\
y' &= -\csc x \cot x + \csc^2 x
\end{aligned}
$$

The simplified form of a derivative after differentiation can be obtained as follows. Notice that the two characteristics of the form are the absence of negative exponents and the combining of like terms.

	$f'(x)$ after differentiating	$f'(x)$ after simplifying
Example 1	$(3x-2x^2)(4)+(5+4x)(3-4x)$	$-24x^2+4x+15$
Example 3	$(2x)(-\sin x)+(\cos x)(2)-2(\cos x)$	$-2x\sin x$
Example 4	$\frac{(x^2+1)(5)-(5x-2)(2x)}{(x^2+1)^2}$	$\frac{-5x^2+4x+5}{(x^2+1)^2}$
Example 5	$\frac{(x^2+5x)(3)-(3x-1)(2x+5)}{(x^2+5x)^2}$	$\frac{-3x^2+2x+5}{(x^2+5x)^2}$
Example 9	$\frac{(\sin x)(\sin x)-(1-\cos x)(\cos x)}{\sin^2 x}$	$\frac{1-\cos x}{\sin^2 x}$

Higher-order derivatives

- You can define derivatives of any positive integer order. For instance, the second derivative is the derivative of the first derivative.
- Higher-order derivatives are denoted as follows.

First derivative:	y',	$f'(x)$,	$\frac{\mathrm{d}y}{\mathrm{d}x}$,	$\frac{\mathrm{d}}{\mathrm{d}x}\,[f(x)]$,	$D_x[y]$
Second derivative:	y'',	$f''(x)$,	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$,	$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,[f(x)]$,	$D_x^2[y]$
Third derivative:	y''',	$f'''(x)$,	$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$,	$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\,[f(x)]$,	$D_x^3[y]$
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{\mathrm{d}^4y}{\mathrm{d}x^4}$,	$\frac{\mathrm{d}^4}{\mathrm{d}x^4}\,[f(x)]$,	$D_x^4[y]$
	$\vdots$				
nth derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{\mathrm{d}^ny}{\mathrm{d}x^n}$,	$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,[f(x)]$,	$D_x^n[y]$

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The Chain Rule

- This text has yet to discuss one of the most powerful differentiation rules—the Chain Rule. This rule deals with composite functions and adds a surprising versatility to the rules discussed in the two previous sections.
- For example, compare the functions shown below. Those on the left can be differentiated without the Chain Rule, and those on the right are best differentiated with the Chain Rule.

Without the Chain Rule	With the Chain Rule
$y = x^2 + 1$	$y = \sqrt{x^2 + 1}$
$y = \sin x$	$y = \sin 6x$
$y = 3x + 2$	$y = (3x + 2)^5$
$y = x + \tan x$	$y = x + \tan x^2$

- Basically, the Chain Rule states that if y changes $\frac{dy}{du}$ times as fast as u, and u changes $\frac{du}{dx}$ times as fast as x, then y changes $\left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$ times as fast as x.

Theorem 2.10 (The Chain Rule)

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

or, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x).$$

- Let $h(x) = f(g(x))$. Then, using the alternative form of the derivative, you need to show that, for $x = c$, $h'(c) = f'(g(c))g'(c)$.
- An important consideration in this proof is the behavior of g as x approaches c. A problem occurs if there are values of x, other than c, such that $g(x) = g(c)$. In the proofs of the Product Rule and the Quotient Rule, the same quantity was added and subtracted to obtain the desired form.

- This proof uses a similar technique-multiplying and dividing by the same (nonzero) quantity.

$$
\begin{aligned}
h'(c) &= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} \\
&= \lim_{x \to c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \times \frac{g(x) - g(c)}{x - c} \right], \quad g(x) \neq g(c)
\end{aligned}
$$

If g oscillates near c, the above expression is undefined because it involves division by zero. To work around this, introduce a function Q as follows:

$$
Q(y) = \begin{cases} \frac{f(y) - f(g(c))}{y - g(c)}, & y \neq g(c) \\ f'(g(c)), & y = g(c) \end{cases}.
$$

We would like to show that

$$
\frac{f(g(x)) - f(g(c))}{x - c} = \left[Q(g(x)) \times \frac{g(x) - g(c)}{x - c} \right]
$$

- Whenever $g(x)$ is not equal to $g(c)$, this is clear because the factors of $g(x) - g(c)$ cancel. When $g(x)$ equals $g(c)$, then the $h'(c)$ is zero because $f(g(x))$ equals $f(g(c))$, and the above product is zero because it equals $f'(g(c))$ times zero. So the above equation is valid. Next, we need to show that the limit as x goes to c of the above product exists and its value equals to $f'(g(c))g'(c)$.
- Notice that the second factor g is differentiable at c by assumption, its limit as x tends to c exists and equals $g'(c)$.
- For $Q(g(x))$, notice that Q is defined wherever f is. Furthermore, f is differentiable at $g(c)$ by assumption, so Q is continuous at $g(c)$, by definition of the derivative of f. Therefore $Q \circ g$ is continuous at c by the rule of composite function. So its limit as x goes to c exists and equals $Q(g(c))$, which is $f'(g(c))$.
- This shows that the limits of both factors exist and therefore, by the product rule the derivative of $f \circ g$ at c exists and equals $f'(g(c))g'(c)$. □

Example 2 (Decomposition of a composite function)

$f(g(x))$	$u = g(x)$	$y = f(u)$
a. $y = \frac{1}{x+1}$	$u = x + 1$	$y = \frac{1}{u}$
b. $y = \sin 2x$	$u = 2x$	$y = \sin u$
c. $y = \sqrt{3x^2 - x + 1}$	$u = 3x^2 - x + 1$	$y = \sqrt{u}$
d. $y = \tan^2 x$	$u = \tan x$	$y = u^2$

■

Example 3 (Applying the chain Rule)

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $y = (x^2 + 1)^3$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3(x^2 + 1)^2(2x) = 6x(x^2 + 1)^2$$

■

The General Power Rule

- One of the most common types of composite functions, $y = [u(x)]^n$.
- The rule for differentiating such functions is called the General Power Rule, and it is a special case of the Chain Rule.

Theorem 2.11 (The General Power Rule)

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = n[u(x)]^{n-1}\frac{\mathrm{d}u}{\mathrm{d}x}$$

or, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}x}[u^n] = nu^{n-1}u'.$$

- Because $y = u^n$, you apply the Chain Rule to obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}u}[u^n]\frac{\mathrm{d}u}{\mathrm{d}x}.$$

- By the (Simple) Power Rule in Section 2, you have $D_u[u^n] = nu^{n-1}$, and it follows that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = n[u(x)]^{n-1}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

□

Example 4 (Applying the general Power Rule)

Find the derivative of $f(x) = (3x - 2x^2)^3$.

- Let $u = 3x - 2x^2$.
- Then $f(x) = (3x - 2x^2)^3 = u^3$ and, by the General Power Rule, the derivative is

$$f'(x) = 3(3x - 2x^2)^2\frac{\mathrm{d}}{\mathrm{d}x}\left[3x - 2x^2\right] = 3(3x - 2x^2)^2(3 - 4x). \quad \blacksquare$$

Example 5 (Differentiating functions involving radicals)

Find all points on the graph of $f(x) = \sqrt[3]{(x^2-1)^2}$ for which $f'(x) = 0$ and those for which $f'(x)$ does not exist.

$$f(x) = (x^2-1)^{2/3} \qquad f'(x) = \frac{2}{3}(x^2-1)^{-1/3}(2x) = \frac{4x}{3\sqrt[3]{x^2-1}}$$

So $f'(x) = 0$ when $x = 0$ and $f'(x)$ does not exist when $x = \pm 1$. ■

Example 6 (Differentiating quotients with constant numerators)

Differentiate $g(t) = \frac{-7}{(2t-3)^2}$.

$$g(t) = -7(2t-3)^{-2} \quad g'(t) = (-7)(-2)(2t-3)^{-3}(2) = 28(2t-3)^{-3}$$

■

- The next three examples illustrate some techniques for simplifying the "raw derivatives" of functions involving products, quotients, and composites.

Example 7 (Simplifying by factoring out the least powers)

Find the derivative of $f(x) = x^2\sqrt{1-x^2}$.

$$
\begin{aligned}
f(x) &= x^2\sqrt{1-x^2} = x^2(1-x^2)^{1/2} \\
f'(x) &= x^2 \frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)^{1/2}\right] + (1-x^2)^{1/2}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\right] \\
&= x^2\left[\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right] + (1-x^2)^{1/2}(2x) \\
&= -x^3(1-x^2)^{-1/2} + 2x(1-x^2)^{1/2} \\
&= x(1-x^2)^{-1/2}\left[-x^2(1) + 2(1-x^2)\right] = \frac{x(2-3x^2)}{\sqrt{1-x^2}} \qquad \blacksquare
\end{aligned}
$$

Example 8 (Simplifying the derivative of a quotient)

$$f(x) = \frac{x}{\sqrt[3]{x^2+4}} = \frac{x}{(x^2+4)^{1/3}}$$

$$f'(x) = \frac{(x^2+4)^{1/3}(1) - x(1/3)(x^2+4)^{-2/3}(2x)}{(x^2+4)^{2/3}}$$

$$= \frac{1}{3}(x^2+4)^{-2/3}\left[\frac{3(x^2+4)-(2x^2)(1)}{(x^2+4)^{2/3}}\right] = \frac{x^2+12}{3(x^2+4)^{4/3}} \quad \blacksquare$$

Example 9 (Simplifying the derivative of a power)

$$y = \left(\frac{3x-1}{x^2+3}\right)^2$$

$$y' = 2\left(\frac{3x-1}{x^2+3}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{3x-1}{x^2+3}\right] = \left[\frac{2(3x-1)}{x^2+3}\right]\left[\frac{(x^2+3)(3)-(3x-1)(2x)}{(x^2+3)^2}\right]$$

$$= \frac{2(3x-1)(3x^2+9-6x^2+2x)}{(x^2+3)^3} = \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3} \quad \blacksquare$$

Example 10 (Applying the Chain Rule to trigonometric functions)

a. $y = \sin \underbrace{2x}_{u}$ $\qquad y' = \underbrace{\cos 2x}_{\cos u} \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}[2x]}_{u'} = (\cos 2x)(2) = 2\cos 2x$

b. $y = \cos(x-1)$ $\qquad y' = -\sin(x-1)$

c. $y = \tan 3x$ $\qquad y' = 3\sec^2 3x$

■

Example 11 (Parentheses and trigonometric functions)

a. $y = \cos 3x^2 = \cos(3x^2)$ $\qquad y' = (-\sin 3x^2)(6x) = -6x \sin 3x^2$

b. $y = (\cos 3)x^2$ $\qquad y' = (\cos 3)(2x) = 2x\cos 3$

c. $y = \cos(3x)^2 = \cos(9x^2)$ $\qquad y' = (-\sin 9x^2)(18x) = -18x \sin 9x^2$

d. $y = \cos^2 x = (\cos x)^2$ $\qquad y' = 2(\cos x)(-\sin x) = -2\cos x \sin x$

e. $y = \sqrt{\cos x} = (\cos x)^{1/2}$ $\qquad y' = \frac{1}{2}(\cos x)^{-1/2}(-\sin x) = -\frac{\sin x}{2\sqrt{\cos x}}$

■

Example 12 (Repeated application of the Chain Rule)

$f(t) = \sin^3 4t = (\sin 4t)^3 \quad f'(t) = 3(\sin 4t)^2(\cos 4t)(4) = 12\sin^2 4t \cos 4t$ ■

Example 13 (Tangent line of a trigonometric function)

Find an equation of the tangent line to the graph of $f(x) = 2\sin x + \cos 2x$ at the point $(\pi, 1)$. Then determine all values of x in the interval $(0, 2\pi)$ at which the graph of f has a horizontal tangent.

$$f(x) = 2\sin x + \cos 2x$$
$$f'(x) = 2\cos x - 2\sin 2x = 2\cos x(1 - 2\sin x)$$
$$f'(\pi) = 2\cos \pi - 2\sin 2\pi = -2$$
$$f'(x) = 0, \qquad x = \frac{\pi}{6},\ \frac{\pi}{2},\ \frac{5\pi}{6},\ \frac{3\pi}{2}$$
$$(y - 1) = -2(x - \pi)$$

horizontal tangent at $x = \frac{\pi}{6},\ \frac{\pi}{2},\ \frac{5\pi}{6},\ \frac{3\pi}{2}$ ■

Table 1: Summary of differentiation rules

General Differentiation Rules	Let u, v be differentiable functions of x	Let f be a differentiable function of u.
	Constant Rule:	(Simple) Power Rule:
	$\frac{d}{dx}[c] = 0$	$\frac{d}{dx}[x^n] = nx^{n-1}$, $\frac{d}{dx}[x] = 1$
	Constant Multiple Rule:	Sum or Difference Rule:
	$\frac{d}{dx}[cu] = cu'$	$\frac{d}{dx}[u \pm v] = u' \pm v'$
	Product Rule:	Quotient Rule:
	$\frac{d}{dx}[uv] = u'v + uv'$	$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v - uv'}{v^2}$
	Chain Rule:	General Power Rule:
	$\frac{d}{dx}[f(u)] = f'(u)u'$	$\frac{d}{dx}[u^n] = nu^{n-1}u'$
Derivatives of Trigonometric Functions	$\frac{d}{dx}[\sin x] = \cos x$ $\frac{d}{dx}[\tan x] = \sec^2 x$ $\frac{d}{dx}[\sec x] = \tan x \sec x$ $\frac{d}{dx}[\cos x] = -\sin x$ $\frac{d}{dx}[\cot x] = -\csc^2 x$ $\frac{d}{dx}[\csc x] = -\cot x \csc x$	

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Implicit and explicit functions

- Most functions have been expressed in explicit form. For example, in the equation

$$y = 3x^2 - 5 \qquad \text{Explicit form}$$

the variable y is explicitly written as a function of x.

- Some functions, however, are only implied by an equation. For instance, the function $y = 1/x$ is defined implicitly by the equation $xy = 1$. Suppose you were asked to find $\mathrm{d}y/\mathrm{d}x$ for this equation. You could begin by writing y explicitly as a function of x and then differentiating.

Implicit Form	Explicit Form	Derivative
$xy = 1$	$y = \frac{1}{x} = x^{-1}$	$\frac{\mathrm{d}y}{\mathrm{d}x} = -x^{-2} = -\frac{1}{x^2}$

- This strategy works whenever you can solve for the function explicitly. You cannot, however, use this procedure when you are unable to solve for y as a function of x.

- For instance, how would you find $\mathrm{d}y/\mathrm{d}x$ for the equation

$$x^2 - 2y^3 + 4y = 2$$

where it is very difficult to express y as a function of x explicitly? To do this, you can use implicit differentiation.

- To understand how to find $\mathrm{d}y/\mathrm{d}x$ implicitly, you must realize that the differentiation is taking place with respect to x. This means that when you differentiate terms involving x alone, you can differentiate as usual.
- However, when you differentiate terms involving y, you must apply the Chain Rule, because you are assuming that y is defined implicitly as a differentiable function of x.

Example 1 (Differentiating with respect to x)

a. $\frac{d}{dx}\left[x^3\right] = 3x^2$ *Variables agree: use Simple Power Rule.*

b. $\frac{d}{dx}\left[y^3\right] = 3y^2\frac{dy}{dx}$ *Variables disagree: use Chain Rule.*

c. $\frac{d}{dx}[x + 3y] = 1 + 3\frac{dy}{dx}$ *Chain Rule:* $\frac{d}{dx}[3y] = 3y'$

d.

$$\frac{d}{dx}[xy^2] = x\frac{d}{dx}[y^2] + y^2\frac{d}{dx}[x] = x\left(2y\frac{dy}{dx}\right) + y^2(1) = 2xy\frac{dy}{dx} + y^2$$

■

Guidelines for implicit differentiation

1. Differentiate both sides of the equation with respect to x.
2. Collect all terms involving $\mathrm{d}y/\mathrm{d}x$ on the left side of the equation and move all other terms to the right side of the equation.
3. Factor $\mathrm{d}y/\mathrm{d}x$ out of the left side of the equation.
4. Solve for $\mathrm{d}y/\mathrm{d}x$.

Example 2 (Implicit differentiation)

Find $\mathrm{d}y/\mathrm{d}x$ given that $y^3 + y^2 - 5y - x^2 = -4$.

- Differentiate both sides of the equation with respect to x.

$$\frac{\mathrm{d}}{\mathrm{d}x}[y^3 + y^2 - 5y - x^2] = \frac{\mathrm{d}}{\mathrm{d}x}[-4]$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[y^3] + \frac{\mathrm{d}}{\mathrm{d}x}[y^2] - \frac{\mathrm{d}}{\mathrm{d}x}[5y] - \frac{\mathrm{d}}{\mathrm{d}x}[x^2] = \frac{\mathrm{d}}{\mathrm{d}x}[-4]$$

$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 5\frac{\mathrm{d}y}{\mathrm{d}x} - 2x = 0$$

- Collect the $\mathrm{d}y/\mathrm{d}x$ terms on the left side of the equation and move all other terms to the right side of the equation.

$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 5\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

- Factor $\mathrm{d}y/\mathrm{d}x$ out of the left side of the equation.

$$\frac{\mathrm{d}y}{\mathrm{d}x}(3y^2 + 2y - 5) = 2x$$

- Solve for $\mathrm{d}y/\mathrm{d}x$ by dividing by $3y^2 + 2y - 5$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{3y^2 + 2y - 5}$$

■

- To see how you can use an implicit derivative, consider the graph shown in Figure 15.
- From the graph, you can see that y is not a function of x. Even so, the derivative found in Example 2 gives a formula for the slope of the tangent line at a point on this graph. The slopes at several points on the graph are shown below the graph.

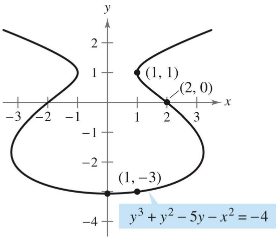

Figure 15: $\mathrm{d}y/\mathrm{d}x$ given that $y^3 + y^2 - 5y - x^2 = -4$.

Example 3 (Representing a graph by differentiable functions)

If possible, represent y as a differentiable function of x.

a. $x^2 + y^2 = 0$ **b.** $x^2 + y^2 = 1$ **c.** $x + y^2 = 1$

a. The graph of this equation is a single point. So, it does not define y as a differentiable function of x.

b. The graph of this equation is the unit circle, centered at $(0, 0)$. The upper semicircle is given by the differentiable function

$$y = \sqrt{1 - x^2}, \quad -1 < x < 1$$

and the lower semicircle is given by the differentiable function

$$y = -\sqrt{1 - x^2}, \quad -1 < x < 1.$$

c. The upper half of this parabola is given by the differentiable function

$$y = \sqrt{1 - x}, \quad x < 1$$

and the lower half of this parabola is given by the differentiable function

$$y = -\sqrt{1 - x}, \quad x < 1.$$

■

Example 4 (Finding the slope of a graph implicitly)

Determine the slope of the tangent line to the graph of $x^2 + 4y^2 = 4$ at the point $\left(\sqrt{2}, -1/\sqrt{2}\right)$.

$$x^2 + 4y^2 = 4 \qquad 2x + 8y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x}{8y} = \frac{-x}{4y}$$

So, at $\left(\sqrt{2}, -1/\sqrt{2}\right)$, the slope is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\sqrt{2}}{-4/\sqrt{2}} = \frac{1}{2}. \qquad \blacksquare$$

Example 7 (Finding the second derivative implicitly)

Given $x^2 + y^2 = 25$, find $\frac{d^2y}{dx^2}$.

- Differentiating each term with respect to x produces

$$2x + 2y\frac{dy}{dx} = 0 \qquad 2y\frac{dy}{dx} = -2x \qquad \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}.$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{y - x\frac{dy}{dx}}{y^2} \\ &= -\frac{y - x(-x/y)}{y^2} \\ &= -\frac{y^2 + x^2}{y^3} \\ &= -\frac{25}{y^3}\end{aligned}$$

■

Example 8 (Finding a tangent line to a graph)

Find the tangent line to the graph given by $x^2(x^2+y^2)=y^2$ at the point $(\sqrt{2}/2, \sqrt{2}/2)$.

$$\begin{aligned} x^4+x^2y^2-y^2 &= 0 \\ 4x^3+x^2\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)+2xy^2-2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x(2x^2+y^2)}{y(1-x^2)} \end{aligned}$$

At the point $(\sqrt{2}/2, \sqrt{2}/2)$, the slope is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(\sqrt{2}/2)(2(1/2)+(1/2))}{(\sqrt{2}/2)(1-(1/2))} = 3$$

and the equation of the tangent line at this point is

$$y-\frac{\sqrt{2}}{2} = 3\left(x-\frac{\sqrt{2}}{2}\right).$$

■