Chapter 1: Limits and Their Properties

Szu-Chi Chung

Department of Applied Mathematics, National Sun Yat-sen University

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What is calculus?

- Calculus is the mathematics of change. For instance, calculus is the mathematics of velocities, accelerations, tangent lines, slopes, areas, volumes, arc lengths, centroids, curvatures, and a variety of other concepts that have enabled scientists, engineers, and economists to model real-life situations.
- Although precalculus mathematics also deals with velocities, accelerations, tangent lines, slopes, and so on, there is a fundamental difference between precalculus mathematics and calculus.
- Precalculus mathematics is more static, whereas calculus is more dynamic.

Here are some examples.

1. An object traveling at a constant velocity can be analyzed with precalculus mathematics. To analyze the velocity of an accelerating object, you need calculus.
2. The slope of a line can be analyzed with precalculus mathematics. To analyze the slope of a curve, you need calculus.
3. The curvature of a circle is constant and can be analyzed with precalculus mathematics. To analyze the variable curvature of a general curve, you need calculus.
4. The area of a rectangle can be analyzed with precalculus mathematics. To analyze the area under a general curve, you need calculus.

So, one way to answer the question "What is calculus?" is to say that calculus is a "limit machine" that involves three stages.

- The first stage is precalculus mathematics, such as the slope of a line or the area of a rectangle.
- The second stage is the limit process, and the third stage is a new calculus formulation, such as a derivative or integral.

$$\boxed{\text{Precalculus mathematics}} \Longrightarrow \boxed{\text{Limit process}} \Longrightarrow \boxed{\text{Calculus}}$$

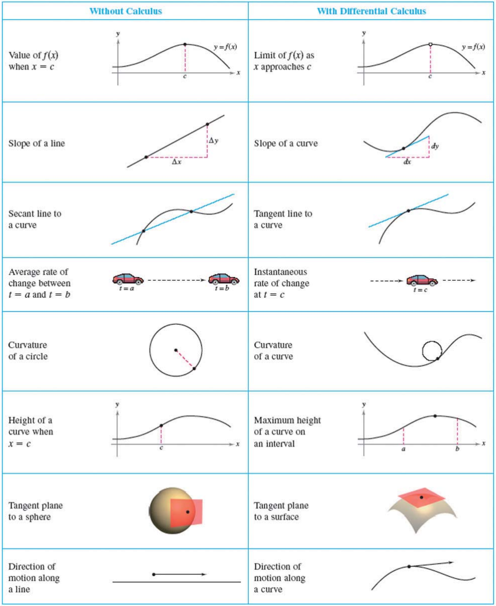


Figure 1: Without calculus versus with differential calculus.

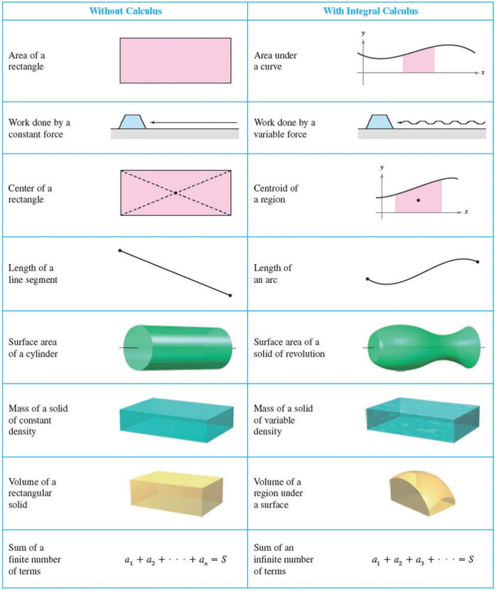


Figure 2: Without calculus versus with integral calculus.

The tangent line problem

- The notion of a limit is fundamental to the study of calculus.
- The following brief descriptions of two classic problems in calculus—tangent line problem and area problem — should give you some idea of the way limits are used in calculus.
- In the tangent line problem, you are given a function f and a point P on its graph and are asked to find an equation of the tangent line to the graph at point P, as shown in Figure 3.

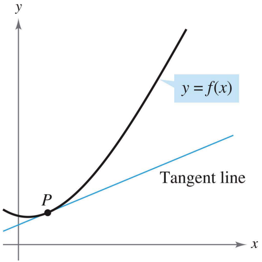
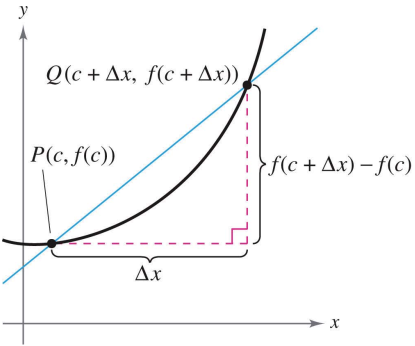


Figure 3: The tangent line to the graph of f at a point.

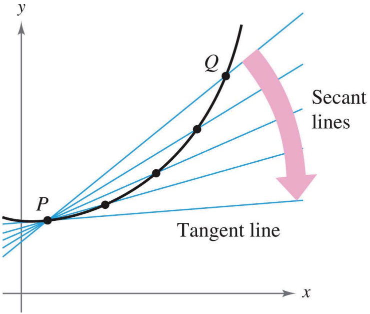
- Except for cases involving a vertical tangent line, the problem of finding the tangent line at a point P is equivalent to finding the slope of the tangent line at P.
- You can approximate this slope by using a line through the point of tangency and a second point on the curve, as shown in Figure 4a. Such a line is called a secant line.
- If $P(c, f(c))$ is the point of tangency and $Q(c+\Delta x, f(c+\Delta x))$ is a second point on the graph of f, then the slope of the secant line through these two points can be found using precalculus and is given by

$$m_{\text{sec}} = \frac{f(c+\Delta x) - f(c)}{c + \Delta x - c} = \frac{f(c+\Delta x) - f(c)}{\Delta x}.$$

- As point Q approaches point P, the slopes of the secant lines approach the slope of the tangent line, as shown in Figure 4b.



(a) The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$.



(b) As Q approaches P, the secant lines approach the tangent line.

Figure 4: The secant line and tangent line.

The area problem

- A second classic problem in calculus is finding the area of a plane region that is bounded by the graphs of functions. This problem can also be solved with a limit process.
  - In this case, the limit process is applied to the area of rectangles to find the area of a general region.
- As a simple example, consider the region bounded by the graph of the function $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$, as shown in Figure 5.

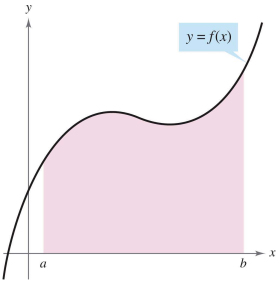
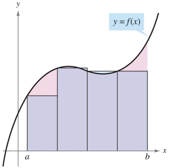
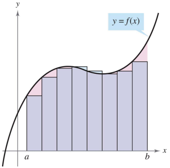


Figure 5: Area under a curve.

- You can approximate the area of the region with several rectangular regions as shown in Figure 6.
- As you increase the number of rectangles, the approximation tends to become better and better because the amount of area missed by the rectangles decreases.
- Your goal is to determine the limit of the sum of the areas of the rectangles as the number of rectangles increases without bound.



(a) Approximation using four rectangles.



(b) Approximation using eight rectangles.

Figure 6: Approximation area under a curve using rectangles.

Notes

- In one of the most astounding events ever to occur in mathematics, it was discovered that the tangent line problem and the area problem are closely related.
- $lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x}$.
- $lim_{n \to \infty} \sum_{j=1}^{n} f(x_j)\Delta x_j$.
- This discovery led to the birth of calculus. You will learn about the relationship between these two problems when you study the Fundamental Theorem of Calculus in Chapter 4.

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An introduction to limits

- Suppose you are asked to sketch the graph of the function f given by

$$f(x) = \frac{x^3 - 1}{x - 1}, \quad x \neq 1.$$

- For all values other than $x = 1$, you can use standard curve-sketching techniques.
- However, at $x = 1$, it is not clear what to expect.
- To get an idea of the behavior of the graph of f near $x = 1$, you can use two sets of x-values—one set that approaches 1 from the left and one set that approaches 1 from the right, as shown in the table.

x approaches 1 from the left. → ← x approaches 1 from the right.

x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.710	2.970	2.997	?	3.003	3.030	3.310	3.813

$f(x)$ approaches 3. → ← $f(x)$ approaches 3.

- The graph of f is a parabola that has a gap at the point $(1, 3)$.
  - Although x can not equal 1, you can move arbitrarily close to 1, and as a result $f(x)$ moves arbitrarily close to 3.
  - Using limit notation, you can write

$$\lim_{x \to 1} f(x) = 3.$$

  - This is read as "the limit of $f(x)$ as x approaches 1 is 3."

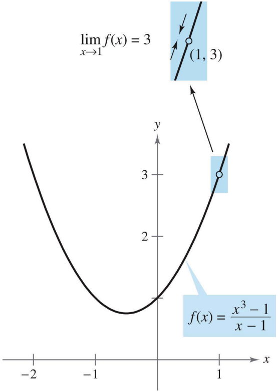


Figure 7: The limit of $f(x) = \frac{x^3-1}{x-1}$ as x approaches 1 is 3.

- This discussion leads to an informal definition of limit.
- If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, the limit of $f(x)$, as x approaches c, is L.
- This limit is written as $\boxed{\lim_{x\to c} f(x) = L}$.

Example 1 (Estimating a limit numerically)

Evaluate the function $f(x) = x/(\sqrt{x+1} - 1)$ at several points near $x = 0$ and use the results to estimate the limit

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}.$$

x approaches 0 from the left. → ← *x* approaches 0 from the right.

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99950	1.99995	?	2.00005	2.00050	2.00499

$f(x)$ approaches 2. → ← $f(x)$ approaches 2.

- The table lists the values of $f(x)$ for several x-values near 0.
- From the results shown in the table, you can estimate the limit to be 2.
- This limit is reinforced by the graph of f (see Figure 8.) ■

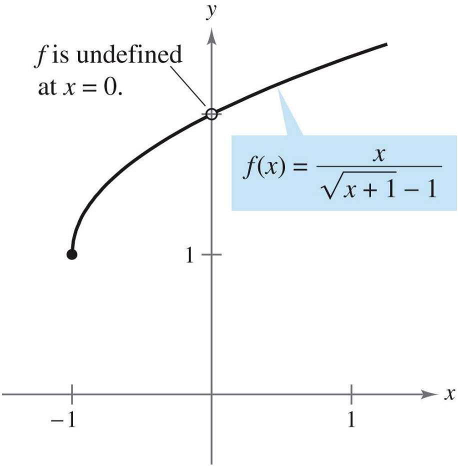


Figure 8: The limit of $f(x) = \frac{x}{\sqrt{x+1}-1}$ as x approaches 0 is 2.

Example 2 (Finding a limit)

Find the limit of $f(x)$ as x approaches 2, where f is defined as

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}.$$

- Because $f(x) = 1$ for all x other than $x = 2$, you can conclude that the limit is 1. So, you can write $\lim_{x \to 2} f(x) = 1$.
- $f(2) = 0$ has no bearing on the existence or value of the limit as x approaches 2.
- For instance, if the function were defined as

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 2, & x = 2 \end{cases}$$

the limit would be the same. ■

Limits that fail to exist

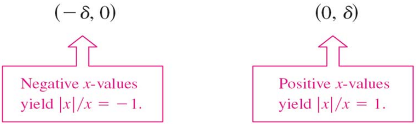
Example 3 (Behavior that differs from the right and from the left)

Show that the limit $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

- From Figure 9 and the definition of $|x|$ you can see that

$$\frac{|x|}{x} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}.$$

- This means that no matter how close x gets to 0, there will be both positive and negative x-values that yield $f(x) = 1$ or $f(x) = -1$.
- Specifically, if δ is a positive number, then for x-values satisfying the inequality $0 < |x| < \delta$, you can classify the values of $|x|/x$ as follows.



- Because $|x|/x$ approaches a different number from the right side of 0 than it approaches from the left side, the limit $\lim_{x\to 0} |x|/x$ does not exist. ∎

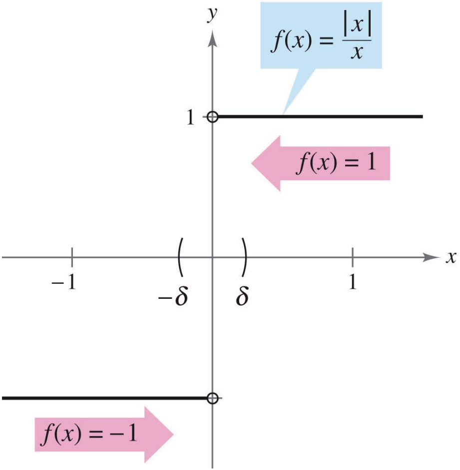


Figure 9: $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Example 4 (Unbounded behavior)

Discuss the existence of the limit $\lim_{x\to 0} \frac{1}{x^2}$.

- Let $f(x) = \frac{1}{x^2}$. In Figure 10, you can see that as x approaches 0 from either the right or the left, $f(x)$ increases without bound.
- This means that by choosing x close enough to 0, you can force $f(x)$ to be as large as you want. For instance, $f(x)$ will be larger than 100 if you choose x that is within $1/10$ of 0. That is,

$$0 < |x| < \frac{1}{10} \implies f(x) = \frac{1}{x^2} > 100.$$

- Similarly, you can force $f(x)$ to be larger than $1,000,000$, as follows.

$$0 < |x| < \frac{1}{1000} \implies f(x) = \frac{1}{x^2} > 1,000,000$$

- Because $f(x)$ is not approaching a real number L as x approaches 0, you can conclude that the limit does not exist. ■

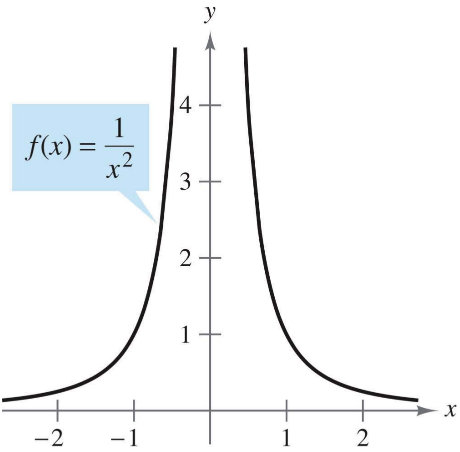


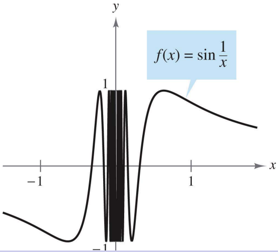
Figure 10: $\lim_{x\to 0} 1/x^2$ does not exist.

Example 5 (Oscillating behavior)

Discuss the existence of the limit $\lim_{x \to 0} \sin \frac{1}{x}$.

- Let $f(x) = \sin(1/x)$. You can see that as x approaches 0, $f(x)$ oscillates between -1 and 1.
- So, the limit does not exist because no matter how small you choose δ, it is possible to choose x_1 and x_2 within δ units of 0 such that $\sin(1/x_1) = 1$ and $\sin(1/x_2) = -1$, as shown in the table. ■

x	$2/\pi$	$2/3\pi$	$2/5\pi$	$2/7\pi$	$2/9\pi$	$2/11\pi$	$x \to 0$
$\sin(1/x)$	1	-1	1	-1	1	-1	Limit does not exist.



Common types of behavior associated with nonexistence of a limit

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

There are many other interesting functions that have unusual limit behavior. An often cited one is the Dirichlet function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational.} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

Because this function has no limit at any real number c, it is actually not continuous at any real number c.

A formal definition of limit

- If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L, is written as

$$\lim_{x \to c} f(x) = L.$$

- At first glance, this definition looks fairly technical. Even so, it is informal because exact meanings have not yet been given to the two phrases "$f(x)$ becomes arbitrarily close to L" and "x approaches c."
- In Figure 12, let ε represent a (small) positive number. Then the phrase "$f(x)$ becomes arbitrarily close to L" means that $f(x)$ lies in the interval $(L - \varepsilon, L + \varepsilon)$. Using absolute value, you can write this as

$$|f(x) - L| < \varepsilon.$$

- Similarly, the phrase "x approaches c" means that there exists a positive number δ such that x lies in either the interval $(c-\delta, c)$ or the interval $(c, c+\delta)$. This fact can be concisely expressed by

$$0 < |x-c| < \delta.$$

  - The first inequality $0 < |x-c|$ says that the distance between x and c is more than 0 which expresses the fact that $x \neq c$. The second inequality $|x-c| < \delta$ indicate that x is within δ units of c.

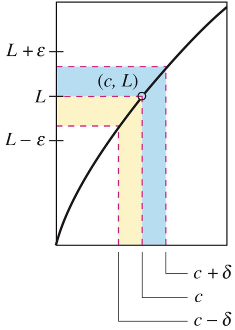


Figure 12: The ε-δ definition of the limit of $f(x)$ as x approaches c.

Definition 1.1 (Limit)

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

Example 6 (Finding a δ for a given ε, Hard)

Given the limit

$$\lim_{x \to 3} (2x - 5) = 1$$

find δ such that $|(2x-5) - 1| < 0.01$ whenever $0 < |x - 3| < \delta$.

- In this problem, you are working with a given value of ε-namely, $\varepsilon = 0.01$. To find an appropriate δ, notice that

$$|(2x-5) - 1| = |2x - 6| = 2|x - 3|.$$

- Because the inequality $|(2x-5) - 1| < 0.01$ is equivalent to $2|x-3| < 0.01$, you can choose $\delta = \frac{1}{2}(0.01) = 0.005$.
- This choice works because $0 < |x - 3| < 0.005$ implies that

$$|(2x-5) - 1| = 2|x - 3| < 2(0.005) = 0.01$$

as shown in Figure 13.

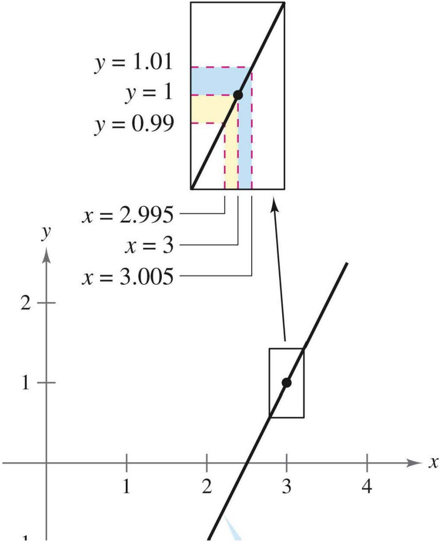


Figure 13: The limit of $f(x) = 2x - 5$ as x approaches 3 is 1.

Example 7 (Using the ε-δ definition of limit, Hard)

Use the ε-δ definition of limit to prove that

$$\lim_{x \to 2} (3x - 2) = 4.$$

- You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|(3x-2)-4| < \varepsilon$ whenever $0 < |x-2| < \delta$.
- Because your choice of δ depends on ε, you need to establish a connection between the absolute values $|(3x-2)-4|$ and $|x-2|$.

$$|(3x-2)-4| = |3x-6| = 3\,|x-2|\,.$$

- So, for a given $\varepsilon > 0$ you can choose $\delta = \varepsilon/3$. This choice works because

$$0 < |x-2| < \delta = \varepsilon/3$$

implies that

$$|(3x-2)-4| = 3\,|x-2| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

Example 8 (Using the ε-δ definition of limit, Hard)

Use the ε-δ definition of limit to prove that

$$\lim_{x \to 2} x^2 = 4.$$

- You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\left|x^2 - 4\right| < \varepsilon$ whenever $0 < |x - 2| < \delta$.
- To find an appropriate δ, begin by writing $\left|x^2 - 4\right| = |x - 2|\,|x + 2|$.
- A useful technique is to first assume that $\delta \leq 1$. For all x in the interval $(1, 3)$, $x + 2 < 5$ and thus $|x + 2| < 5$.
- So, letting δ be the minimum of $\varepsilon/5$ and 1, it follows that, whenever $0 < |x - 2| < \delta$, you have

$$\left|x^2 - 4\right| = |x - 2|\,|x + 2| < \left(\frac{\varepsilon}{5}\right)(5) = \varepsilon.$$

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Properties of limits

- The limit of $f(x)$ as x approaches c does not depend on the value of f at $x = c$. It may happen, however, that the limit is precisely $f(c)$.
- In such cases, the limit can be evaluated by direct substitution. That is,

$$\lim_{x \to c} f(x) = f(c).$$

Such well-behaved functions are continuous at c.

Theorem 1.1 (Some basic limits)

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$

Example 1 (Evaluating basic limits)

a. $\lim_{x\to 2} 3 = 3$ **b.** $\lim_{x\to -4} x = -4$ **c.** $\lim_{x\to 2} x^2 = 4$ ■

Theorem 1.2 (Properties of limits)

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x\to c} f(x) = L \quad \textit{and} \quad \lim_{x\to c} g(x) = K$$

1. *Scalar multiple:* $\lim_{x\to c}[bf(x)] = bL$
2. *Sum or difference:* $\lim_{x\to c}[f(x) \pm g(x)] = L \pm K$
3. *Product:* $\lim_{x\to c}[f(x)g(x)] = LK$
4. *Quotient:* $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K}$, *provided* $K \neq 0$
5. *Power:* $\lim_{x\to c}[f(x)]^n = L^n$

Theorem 1.3 (Limits of polynomial and rational functions)

If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

Example 2 (The limit of a polynomial)

Find the limit: $\lim_{x \to 2}(4x^2 + 3)$.

- By direct substitution:

$$\lim_{x \to 2}(4x^2 + 3) = \lim_{x \to 2} 4x^2 + \lim_{x \to 2} 3 = 4\left(\lim_{x \to 2} x^2\right) + \lim_{x \to 2} 3$$
$$= 4(2^2) + 3 = 19$$

Example 3 (The limit of a rational function)

Find the limit: $\lim_{x \to 1} \frac{x^2+x+2}{x+1}$.

- Because the denominator is not 0 when $x = 1$, you can apply Theorem 1.3 to obtain

$$\lim_{x \to 1} \frac{x^2 + x + 2}{x + 1} = \frac{1^2 + 1 + 2}{1 + 1} = \frac{4}{2} = 2. \qquad \blacksquare$$

Theorem 1.4 (The limit of a function involving a radical)

Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

Theorem 1.5 (The limit of a composite function)

If f and g are functions such that $\lim_{x\to c} g(x) = L$ *and* $\lim_{x\to L} f(x) = f(L)$*, then*

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(L).$$

Example 4 (The limit of a composite function)

Find the limit.
a. $\lim_{x\to 0}\sqrt{x^2+4}$ **b.** $\lim_{x\to 3}\sqrt[3]{2x^2-10}$

a. Because

$$\lim_{x\to 0}(x^2+4)=0^2+4=4 \quad \text{and} \quad \lim_{x\to 4}\sqrt{x}=\sqrt{4}=2$$

if follows that

$$\lim_{x\to 0}\sqrt{x^2+4}=\sqrt{4}=2.$$

b. Because

$$\lim_{x\to 3}(2x^2-10)=2(3^2)-10=8 \quad \text{and} \quad \lim_{x\to 8}\sqrt[3]{x}=\sqrt[3]{8}=2$$

if follows that

$$\lim_{x\to 3}\sqrt[3]{2x^2-10}=\sqrt[3]{8}=2.$$

■

Theorem 1.6 (Limits of trigonometric functions)

Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x\to c} \sin x = \sin c$
2. $\lim_{x\to c} \cos x = \cos c$
3. $\lim_{x\to c} \tan x = \tan c$
4. $\lim_{x\to c} \cot x = \cot c$
5. $\lim_{x\to c} \sec x = \sec c$
6. $\lim_{x\to c} \csc x = \csc c$

Example 5 (Limits of trigonometric functions)

a. $\lim_{x\to 0} \tan x = \tan(0) = 0$

b. $\lim_{x\to \pi} (x\cos x) = \left(\lim_{x\to\pi} x\right)\left(\lim_{x\to\pi} \cos x\right) = \pi\cos(\pi) = -\pi$

c. $\lim_{x\to 0} \sin^2 x = \lim_{x\to 0} (\sin x)^2 = 0^2 = 0$ ■

A strategy for finding limits

Theorem 1.7 (Functions that agree at all but one point)

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

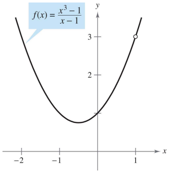
Example 6 (Finding the limit of a function)

Find the limit: $\lim_{x \to 1} \frac{x^3-1}{x-1}$.

- Let $f(x) = (x^3 - 1)/(x - 1)$.
- By factoring and dividing out like factors, you can rewrite f as

$$f(x) = \frac{(x-1)(x^2+x+1)}{(x-1)} = x^2 + x + 1 = g(x), \quad x \neq 1.$$

- So, for all x-values other than $x = 1$, the functions f and g agree, as shown in Figure 14.



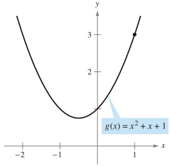


Figure 14: $f(x) = \frac{x^3-1}{x-1}$, $x \neq 1$ and $g(x) = x^2 + x + 1$ agree at all but one point.

- Because $\lim_{x\to 1} g(x)$ exists, you can apply Theorem 1.7 to conclude that f and g have the same limit at $x = 1$.

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{x^2+x+1}{x-1} = \lim_{x\to 1}(x^2+x+1) = 1^2+1+1 = 3$$

■

A strategy for finding limits

1. Learn to recognize which limits can be evaluated by direct substitution. (These limits are listed in Theorems 1.1 through 1.6.)
2. If the limit of $f(x)$ as x approaches c cannot be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$.
3. Apply Theorem 1.7 to conclude analytically that
$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = g(c).$$
4. Use a graph or table to reinforce your conclusion.

Dividing out and rationalizing techniques

- Two techniques for finding limits analytically are shown in Examples 7 and 8.
- The dividing out technique involves dividing out common factors, and the rationalizing technique involves rationalizing the numerator of a fractional expression.

Example 7 (Dividing out technique)

Find the limit: $\lim_{x\to -3} \frac{x^2+x-6}{x+3}$.

- Although you are taking the limit of a rational function, you cannot apply Theorem 1.3 because the limit of the denominator is 0.

$$\lim_{x\to -3} \frac{x^2+x-6}{x+3}$$

$$\Longrightarrow \quad \begin{cases} \lim_{x\to -3}(x^2+x-6)=0 \\ \lim_{x\to -3}(x+3)=0 \end{cases} \qquad \text{Direct substitution fails}$$

- Because the limit of the numerator is also 0, the numerator and denominator have a common factor of $(x+3)$. So, for all $x \neq -3$, you can divide out this factor to obtain

$$f(x) = \frac{x^2+x-6}{x+3} = \frac{(x+3)(x-2)}{x+3} = x-2 = g(x), \quad x \neq -3.$$

- Using Theorem 1.7, it follows that

$$\lim_{x \to -3} \frac{x^2+x-6}{x+3} = \lim_{x \to -3} (x-2) = -5. \qquad \blacksquare$$

- This result is shown graphically in Figure 15.

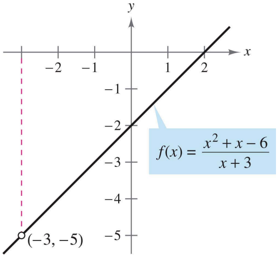


Figure 15: $f(x) = \frac{x^2+x-6}{x+3}$ is undefined when $x = -3$.

- An expression such as $0/0$ is called an indeterminate form because you cannot (from the form alone) determine the limit. When you try to evaluate a limit and encounter this form, remember that you must rewrite the fraction so that the new denominator does not have 0 as its limit.
- One way to do this is to divide out like factors, as shown in Example 7. A second way is to rationalize the numerator, as shown in Example 8.

Example 8 (Rationalizing technique)

Find the limit: $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$.

- By direct substitution, you obtain the indeterminate form $0/0$.

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$$

$$\Longrightarrow \quad \begin{cases} \lim_{x\to 0} \left(\sqrt{x+1}-1\right) = 0 \\ \lim_{x\to 0} x = 0 \end{cases} \qquad \text{Direct substitution fails}$$

- In this case, you can rewrite the fraction by rationalizing the numerator.

$$\frac{\sqrt{x+1}-1}{x} = \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)$$

$$= \frac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)} = \frac{1}{\sqrt{x+1}+1}, \quad x \neq 0$$

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0}\frac{1}{\sqrt{x+1}+1} = \frac{1}{1+1} = \frac{1}{2}$$

- A table or a graph can reinforce your conclusion that the limit is $\frac{1}{2}$. ■

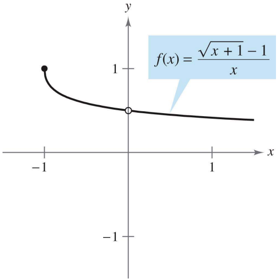


Figure 16: The limit of $f(x) = \frac{\sqrt{x+1}-1}{x}$ as x approaches 0 is $\frac{1}{2}$.

x approaches 0 from the left. → ← x approaches 0 from the right.

x	-0.25	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	0.25
$f(x)$	0.5359	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881	0.4721

$f(x)$ approaches 0.5. → ← $f(x)$ approaches 0.5.

- The next theorem concerns the limit of a function that is squeezed between two other functions, each of which has the same limit at a given x-value, as shown in Figure 17.

$$h(x) \le f(x) \le g(x)$$

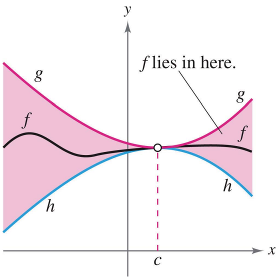


Figure 17: The Squeeze Theorem.

The Squeeze Theorem

Theorem 1.8 (The Squeeze Theorem)

If $h(x) \leq f(x) \leq g(x)$ *for all* x *in an open interval containing* c*, except possibly at* c *itself, and if*

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

then $\lim_{x \to c} f(x)$ *exists and is equal to* L*.*

- Squeeze Theorem is also called the Sandwich Theorem or the Pinching Theorem.

Theorem 1.9 (Two special trigonometric limits)

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$ 2. $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

1. The proof is presented using the variable θ, where θ is an acute positive angle measured in radians. Figure 18 shows a circular sector that is squeezed between two triangles.

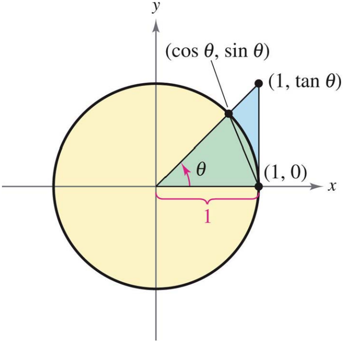
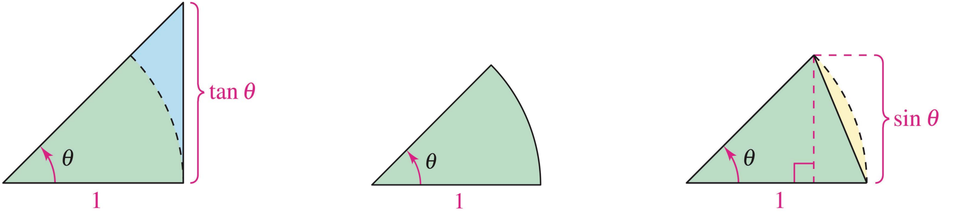


Figure 18: A circular sector is used to prove Theorem 1.9.



Area of triangle $= \frac{\tan\theta}{2}$ $\geq$ Area of sector $= \frac{\theta}{2}$ $\geq$ Area of triangle $= \frac{\sin\theta}{2}$

- Multiplying each expression by $2/\sin\theta$ produces

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1$$

and taking reciprocals and reversing the inequalities yields

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1.$$

- Because $\cos\theta = \cos(-\theta)$ and $(\sin\theta)/\theta = [\sin(-\theta)]/(-\theta)$, you can conclude that this inequality is valid for all nonzero θ in the open interval $(-\pi/2, \pi/2)$.
- Finally, because $\lim_{\theta\to 0}\cos\theta = 1$ and $\lim_{\theta\to 0} 1 = 1$, you can apply the Squeeze Theorem to conclude that $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$.

2.

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x} &= \lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}\\ &= \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x} = \left[\lim_{x\to 0}\frac{\sin x}{x}\right]\left[\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right]\\ &= (1)(0) = 0.\end{aligned}$$

□

Example 9 (A limit involving a trigonometric function)

Find the limit: $\lim_{x\to 0}\frac{\tan x}{x}$.

- Direct substitution yields the indeterminate form $0/0$.
- To solve this problem, you can write $\tan x$ as $(\sin x)/(\cos x)$ and obtain

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right).$$

- Now, because

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \quad \text{and} \quad \lim_{x\to 0}\frac{1}{\cos x}=1$$

you can obtain (See Figure 19.)

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right)=(1)(1)=1. \qquad \blacksquare$$

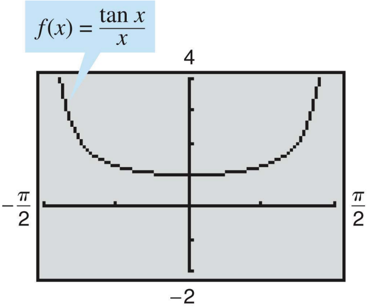


Figure 19: The limit of $f(x)=\frac{\tan x}{x}$ as x approaches 0 is 1.

Example 10 (A limit involving a trigonometric function)

Find the limit: $\lim_{x\to 0} \frac{\sin 4x}{x}$.

$$\lim_{x\to 0} \frac{\sin 4x}{x} = 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right) = 4\left(\lim_{y\to 0} \frac{\sin y}{y}\right) = 4(1) = 4 \qquad \blacksquare$$

Table of Contents

1. A preview of calculus
2. Finding limits graphically and numerically
3. Evaluating limits analytically
4. Continuity and one-sided limits
5. Infinite limits

Continuity at a point and on an open interval

- The term continuous is to say that a function f is continuous at $x = c$ means that there is no interruption in the graph of f at c.
- That is, its graph is unbroken at c and there are no holes, jumps, or gaps. Figure 20 identifies three values of x at which the graph of f is not continuous.

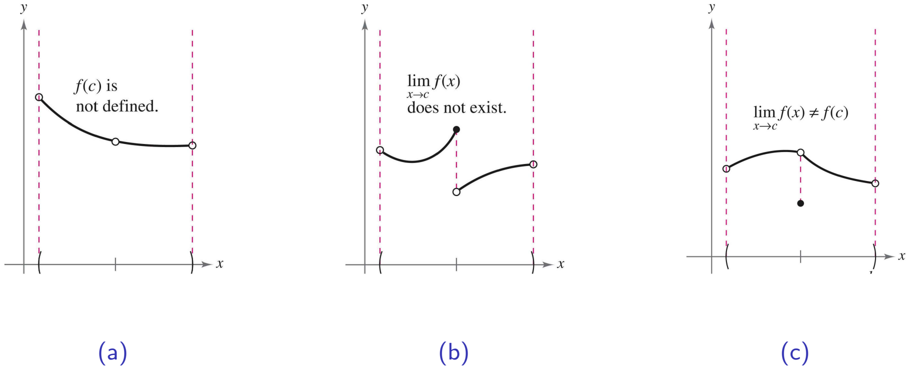


Figure 20: Three conditions the graph of f is not continuous at $x = c$.

- In Figure 20, it appears that continuity at $x = c$ can be destroyed by any one of the following conditions.
  1. The function is not defined at $x = c$.
  2. The limit of $f(x)$ does not exist at $x = c$.
  3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.
- If none of the three conditions above is true, the function f is called continuous at c, as indicated in the following important definition.

Definition 1.2 (Continuity)

Continuity at a point: A function f is continuous at c if the following three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x\to c} f(x)$ exists.
3. $\lim_{x\to c} f(x) = f(c)$

Continuity on an open interval: A function is continuous on an open interval (a, b) if it is continuous at each point in the interval.
Continuity on $\mathbb{R}$: A function that is continuous on the entire real line $(-\infty, \infty)$ is everywhere continuous.

- Consider an open interval I that contains a real number c. If a function f is defined on I (except possibly at c), and f is not continuous at c, then f is said to have a discontinuity at c.
- Discontinuities fall into two categories: removable and nonremovable.
    - A discontinuity at c is called removable if f can be made continuous by appropriately defining (or redefining $f(c)$).
    - For instance, the functions shown in Figures 20(a) and 20(c) have removable discontinuities at c and the function shown in Figure 20(b) has a nonremovable discontinuity at c.

Example 1 (Continuity of a function)

Discuss the continuity of each function. **a.** $f(x) = \frac{1}{x}$ **b.** $g(x) = \frac{x^2-1}{x-1}$

c. $h(x) = \begin{cases} x+1, & x \le 0 \\ x^2+1, & x > 0 \end{cases}$ **d.** $y = \sin x$

a. The domain of f is all nonzero real numbers. From Theorem 1.3, you can conclude that f is continuous at every x-value in its domain. At $x = 0$, f has a non removable discontinuity In other words, there is no way to define $f(0)$ so as to make the function continuous at $x = 0$.

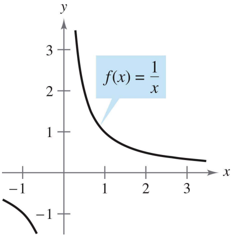


Figure 21: Nonremovable discontinuity of $f(x) = \frac{1}{x}$ at $x = 0$.

b. The domain of g is all real numbers except $x = 1$. From Theorem 1.3, you can conclude that g is continuous at every x-value in its domain. At $x = 1$, the function has a removable discontinuity, as shown in Figure 22. If $g(1)$ is defined as 2, the "newly defined" function is continuous for all real numbers.

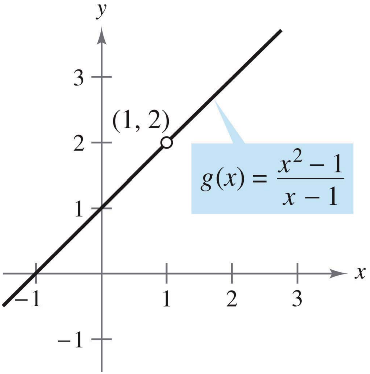


Figure 22: Removable discontinuity of $g(x) = \frac{x^2-1}{x-1}$ at $x = 1$.

c. The domain of h is all real numbers. The function h is continuous on $(-\infty, 0)$ and $(0, \infty)$, and, because $\lim_{x\to 0} h(x) = 1$, h is continuous on the entire real line, as shown in Figure 23.

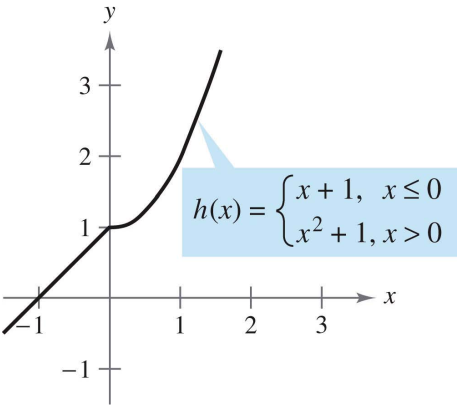


Figure 23: $h(x) = x + 1$, if $x \le 0$ and $x^2 + 1$, $x > 0$ is continuous on entire real line.

d. The domain of y is all real numbers. From Theorem 1.6, you can conclude that the function is continuous on its entire domain, $(-\infty, \infty)$, as shown in Figure 24. ■

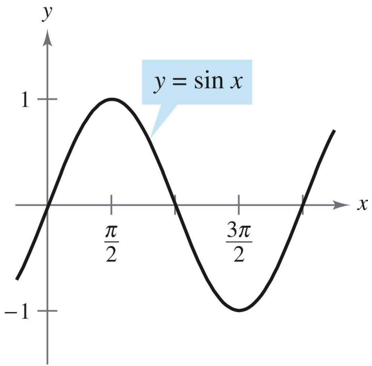
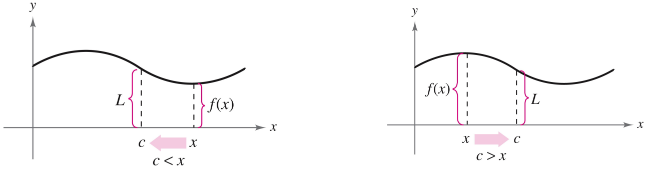


Figure 24: $y = \sin x$ is continuous on entire real line.

One-sided limits and continuity on a closed interval

- To understand continuity on a closed interval, you first need to look at a different type of limit called a one-sided limit.
- For example, the limit from the right (or right-hand limit) means that x approaches c from values greater than c [see Figure 25(a)].



(a) Limit from right.

(b) Limit from left.

Figure 25: One-sided limits.

- This limit is denoted as

$$\lim_{x \to c^+} f(x) = L. \qquad \text{Limit from the right}$$

- Similarly, the limit from the left (or left-hand limit) means that x approaches c from values less than c [see Figure 25(b)].
- This limit is denoted as

$$\lim_{x\to c^-} f(x) = L. \qquad \text{Limit from the left}$$

- One-sided limits are useful in taking limits of functions involving radicals.
- For instance, if n is an even integer,

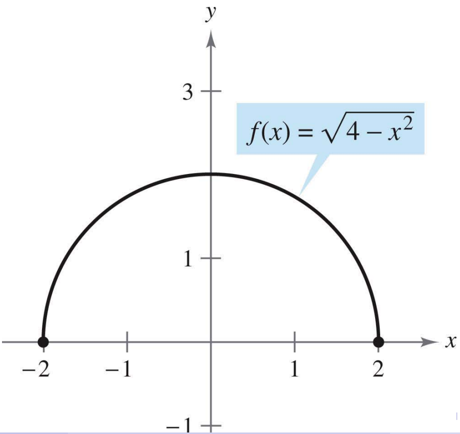
$$\lim_{x\to 0^+} \sqrt[n]{x} = 0.$$

Example 2 (A one-sided limit)

Find the limit of $f(x) = \sqrt{4 - x^2}$ as x approaches -2 from the right.

- As shown in Figure 26, the limit as x approaches -2 from the right is

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0.$$ ■



- One-sided limits can be used to investigate the behavior of step functions.
- One common type of step function is the greatest integer function $\lfloor x \rfloor$, defined by

$$\lfloor x \rfloor = \text{greatest integer } n \text{ such that } n \leq x.$$

- For instance, $\lfloor 2.5 \rfloor = 2$ and $\lfloor -2.5 \rfloor = -3$.

Example 3 (The greatest integer function)

Find the limit of $f(x) = \lfloor x \rfloor$ as x approaches 0 from the left and from the right.

$$\lim_{x \to 0^-} \lfloor x \rfloor = -1$$

$$\lim_{x \to 0^+} \lfloor x \rfloor = 0$$

Theorem 1.10 (The existence of a limit)

Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Definition 1.3 (Continuity on a closed interval)

A function f is continuous on the closed interval $[a, b]$ if it is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b).$$

The function f is continuous from the right at a and continuous from the left at b (see Figure 27).

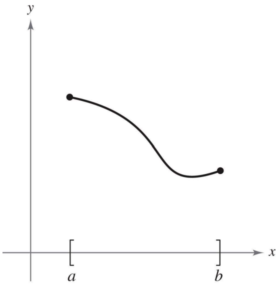


Figure 27: Continuous function on a closed interval.

Example 4 (Continuity on a closed interval)

Discuss the continuity of $f(x) = \sqrt{1 - x^2}$.

- The domain of f is the closed interval $[-1, 1]$. At all points in the open interval $(-1, 1)$, the continuity of f follows from Theorems 1.3 and 1.5.

- Moreover, because

$$\lim_{x\to -1^+} \sqrt{1-x^2} = 0 = f(-1) \qquad \text{Continuous from the right}$$

and

$$\lim_{x\to -1^-} \sqrt{1-x^2} = 0 = f(1) \qquad \text{Continuous from the left}$$

you can conclude that f is continuous on the closed interval $[-1, 1]$, as shown in Figure 28. ■

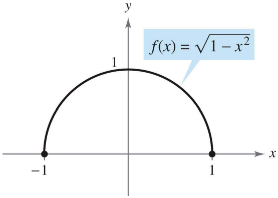


Figure 28: $f(x) = \sqrt{1-x^2}$ is continuous on $[-1, 1]$.

Example 5 (Charles's Law and absolute zero)

On the Kelvin scale, absolute zero is the temperature 0 K. Although temperatures very close to 0 K have been produced in laboratories, absolute zero has never been attained. In fact, evidence suggests that absolute zero cannot be attained. How did scientists determine that 0 K is the "lower limit" of the temperature of matter? What is absolute zero on the Celsius scale?

- To generate the values in the table, one mole of hydrogen is held at a constant pressure of one atmosphere. The volume V is approximated and is measured in liters, and the temperature T is measured in degrees Celsius.
- The points represented by the table are shown in Figure 29. Moreover, by using the points in the table, you can determine that T and V are related by the linear equation

$$V = 0.08213T + 22.4334 \quad \text{or} \quad T = \frac{V - 22.4334}{0.08213}.$$

- By reasoning that the volume of the gas can approach 0 (but can never equal or go below 0), you can determine that the "least possible temperature" is given by

$$\lim_{V\to 0^+} T = \lim_{V\to 0^+} \frac{V - 22.4334}{0.08213} = \frac{0 - 22.4334}{0.08213} \approx -273.15.$$

- So, absolute zero on the Kelvin scale (0 K) is approximately -273.15° on the Celsius scale. ■

T	-40	-20	0	20	40	60	80
V	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.0038

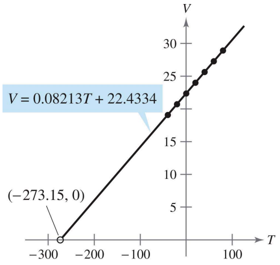


Figure 29: The volume of hydrogen gas depends on its temperature.

Properties of continuity

Theorem 1.11 (Properties of continuity)

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. *Scalar multiple:* bf
2. *Sum or difference:* $f \pm g$
3. *Product:* fg
4. *Quotient:* $\frac{f}{g}$, *if* $g(c) \neq 0$

- The following types of functions are continuous at every point in their domains.
  1. Polynomial: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
  2. Rational: $r(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$
  3. Radical: $f(x) = \sqrt[n]{x}$
  4. Trigonometric: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$
- By combining Theorem 1.11 with this summary, you can conclude that a wide variety of elementary functions are continuous at every point in their domains.

Example 6 (Applying properties of continuity)

By Theorem 1.11, it follows that each of the functions below is continuous at every point in its domain.

$$f(x) = x + \sin x, \quad f(x) = 3\tan x, \quad f(x) = \frac{x^2+1}{\cos x}. \qquad \blacksquare$$

- The next theorem, which is a consequence of Theorem 1.5, allows you to determine the continuity of composite functions such as

$$f(x) = \sin 3x, \quad f(x) = \sqrt{x^2+1}, \quad f(x) = \tan \frac{1}{x}.$$

Theorem 1.12 (Continuity of a composite function)

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

- By the definition of continuity, $\lim_{x \to c} g(x) = g(c)$ and $\lim_{x \to g(c)} = f(g(c))$.
- Apply Theorem 1.5 with $L = g(c)$ to obtain $\lim_{x \to c} f(g(x)) = f(\lim_{x \to c} g(x)) = f(g(c))$. So, $(f \circ g) = f(g(x))$ is continuous at c. $\square$

Example 7 (Testing for continuity)

Describe the interval(s) on which each function is continuous.

a. $f(x) = \tan x$ **b.** $g(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

c. $h(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

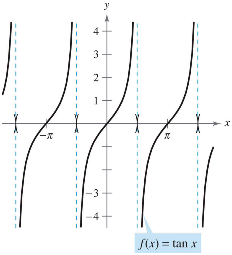
a. The tangent function $f(x) = \tan x$ is undefined at

$$x = \frac{\pi}{2} + n\pi, \quad n \text{ is an integer.}$$

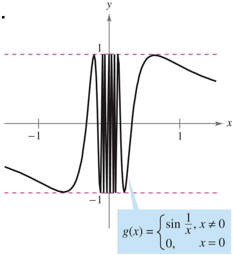
- At all other points it is continuous.
- So, $f(x) = \tan x$ is continuous on the open intervals

$$\ldots, \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$$

as shown below.



b. Because $y = 1/x$ is continuous except at $x = 0$ and the sine function is continuous for all real values of x, it follows that $y = \sin(1/x)$ is continuous at all real values except $x = 0$. At $x = 0$, the limit of $g(x)$ does not exist. So, g is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$ as shown below.



c. This function is similar to the function in part (b) except that the oscillations are damped by the factor x.

- Using the Squeeze Theorem, you obtain

$$-|x| \leq x \sin \frac{1}{x} \leq |x|, \quad x \neq 0$$

and you can conclude that

$$\lim_{x \to 0} h(x) = 0.$$

- So, h is continuous on the entire real line, as shown in Figure 30. ■

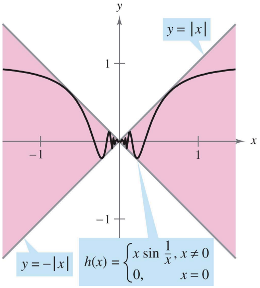


Figure 30: h is continuous on the entire real line.

The Intermediate Value Theorem

A theorem verifying that the graph of a continuous function is connected.

Theorem 1.13 (The Intermediate Value Theorem)

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

- The Intermediate Value Theorem tells you that at least one number c exists, but it does not provide a method for finding c. Such theorems are called <u>existence theorems</u>. A proof of this theorem is based on a property of real numbers called completeness.
- The Intermediate Value Theorem states that for a continuous function f, if x takes on all values between a and b, $f(x)$ must take on all values between $f(a)$ and $f(b)$.

- Suppose that a girl is 160 centimeters tall on her thirteenth birthday and 165 centimeters tall on her fourteenth birthday. Then, for any height h between 160 centimeters and 165 centimeters, there must have been a time t when her height was exactly h. This seems reasonable because human growth is continuous and a person's height does not abruptly change from one value to another.
- The Intermediate Value Theorem guarantees the existence of at least one number c in the closed interval $[a, b]$. There may, of course, be more than one number c such that $f(c) = k$, as shown in Figure 31.

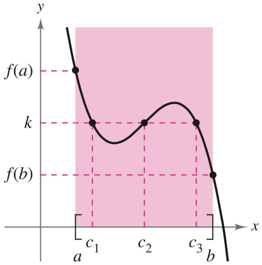


Figure 31: Intermediate Value Theorem: f is continuous on $[a, b]$. (There exists three c's such that $f(c) = k$.)

- A function that is not continuous does not necessarily exhibit the intermediate value property.
- For example, the graph of the function shown in Figure 32 jumps over the horizontal line given by $y = k$, and for this function there is no value of c in $[a, b]$ such that $f(c) = k$.
- The Intermediate Value Theorem often can be used to locate the zeros of a function that is continuous on a closed interval. Specifically, if f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, the Intermediate Value Theorem guarantees the existence of at least one zero of f in the closed interval $[a, b]$.

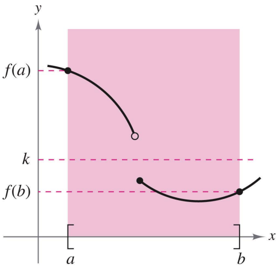


Figure 32: f is not continuous on $[a, b]$.

Example 8 (An application of the Intermediate Value Theorem)

Use the Intermediate Value Theorem to show that the polynomial function $f(x) = x^3 + 2x - 1$ has a zero in the interval $[0, 1]$.

- Note that f is continuous on the closed interval $[0, 1]$. Because

$$f(0) = 0^3 + 2(0) - 1 = -1 \quad \text{and} \quad f(1) = 1^3 + 2(1) - 1 = 2$$

it follows that $f(0) < 0$ and $f(1) > 0$. You can therefore conclude that there must be some c in $[0, 1]$ such that $f(c) = 0$ ■

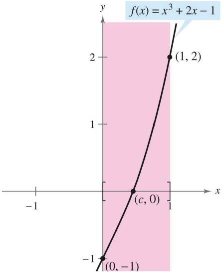


Figure 33: f is continuous on $[0, 1]$ with $f(0) < 0$ and $f(1) > 0$.

- The Bisection Method for approximating the real zeros of a continuous function is similar to the method used in Example 8.
- If you know that a zero exists in the closed interval $[a, b]$, the zero must lie in the interval $[a, (a+b)/2]$ or $[(a+b)/2, b]$.
- From the sign of $f([a+b]/2)$, you can determine which interval contains the zero.
- By repeatedly bisecting the interval, you can "close in" on the zero of the function.

Table of Contents

1. A preview of calculus
2. Finding limits graphically and numerically
3. Evaluating limits analytically
4. Continuity and one-sided limits
5. Infinite limits

Infinite limits

- Let f be the function given by $3/(x-2)$. From Figure 34 and the table, you can see that $f(x)$ decreases without bound as x approaches 2 from the left, and $f(x)$ increases without bound as x approaches 2 from the right.

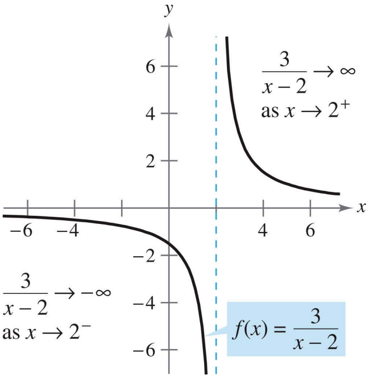


Figure 34: $f(x) = \frac{3}{x-2}$ increases and decreases without bound as x approaches 2.

x approaches 2 from the left. → ← x approaches 2 from the right.

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
$f(x)$	-6	-30	-300	-3000	?	3000	300	30	6

$f(x)$ decreases without bound. → ← $f(x)$ increases without bound.

- This behavior is denoted as

$$\lim_{x \to 2^-} \frac{3}{x-2} = -\infty$$

$f(x)$ decreases without bound as x approaches 2 from the left

and

$$\lim_{x \to 2^+} \frac{3}{x-2} = \infty$$

$f(x)$ increases without bound as x approaches 2 from the right

- A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an infinite limit.

Definition 1.4 (Infinite limit)

Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement

$$\lim_{x \to c} f(x) = \infty$$

means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$ (see Figure 35). Similarly, the statement

$$\lim_{x \to c} f(x) = -\infty$$

means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - c| < \delta$.
To define the infinite limit from the left, replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the infinite limit from the right, replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.

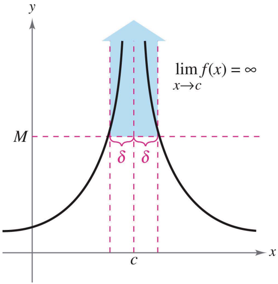


Figure 35: Infinite limits.

Example 1 (Determining infinite limits from a graph)

Determine the limit of each function shown in Figure 36 as x approaches 1 from the left and from the right.

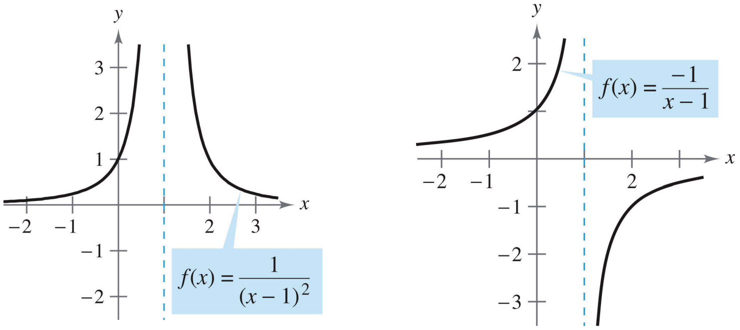


Figure 36: $f(x) = \frac{1}{(x-1)^2}$ and $f(x) = \frac{-1}{x-1}$ have an asymptote at $x = 1$.

(a) When x approaches 1 from the left or the right, $(x-1)^2$ is a small positive number. Thus, the quotient $\frac{1}{(x-1)^2}$ is a large positive number and $f(x)$ approaches infinity from each side of $x = 1$. So, you can conclude that

$$\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty. \qquad \text{Limit from each side is infinity}$$

(b) When x approaches 1 from the left, $x-1$ is a small negative number.

- Thus, the quotient $\frac{-1}{(x-1)}$ is a large positive number and $f(x)$ approaches infinity from left of $x=1$.
- So, you can conclude that

$$\lim_{x\to 1^-}\frac{-1}{(x-1)}=\infty. \qquad \text{Limit from the left side is infinity}$$

When x approaches 1 from the right, $x-1$ is a small positive number.

- Thus, the quotient $\frac{-1}{(x-1)}$ is a large negative number and $f(x)$ approaches negative infinity from the right of $x=1$.
- So, you can conclude that

$$\lim_{x\to 1^+}\frac{-1}{(x-1)}=-\infty. \quad \text{Limit from the right side is negative infinity}$$

■

Vertical asymptotes

Definition 1.5 (Vertical asymptote)

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a vertical asymptote of the graph of f.

Theorem 1.14 (Vertical asymptotes)

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function given by

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.

Example 2 (Find vertical asymptotes)

Determine all vertical asymptotes of the graph of each function. **a.** $f(x) = \frac{1}{2(x+1)}$ **b.** $f(x) = \frac{x^2+1}{x^2-1}$ **c.** $f(x) = \cot x$

a. When $x = -1$, the denominator of $f(x) = \frac{1}{2(x+1)}$ is 0 and the numerator is not 0.

- So, by Theorem 1.14, you can conclude that $x = -1$ is a vertical asymptote, as shown in Figure 37.

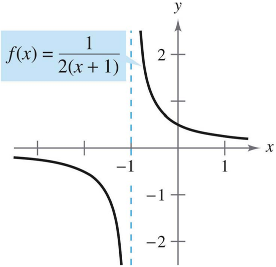


Figure 37: $f(x) = \frac{1}{2(x+1)}$ has an asymptote at $x = -1$.

b. By factoring the denominator as

$$f(x) = \frac{x^2+1}{x^2-1} = \frac{x^2+1}{(x-1)(x+1)}$$

you can see that the denominator is 0 at $x = -1$ and $x = 1$.

- Moreover, because the numerator is not 0 at these two points, you can apply Theorem 1.14 to conclude that the graph of f has two vertical asymptotes, as shown in Figure 38.

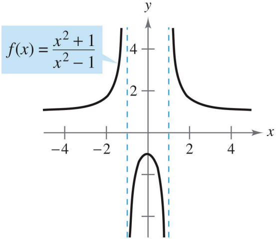


Figure 38: $f(x) = \frac{x^2+1}{x^2-1}$ has vertical asymptotes at $x = \pm 1$.

c. By writing the cotangent function in the form

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

you can apply Theorem 1.14 to conclude that vertical asymptotes occur at all values of x such that $\sin x = 0$ and $\cos x \neq 0$, as shown in Figure 39.

- So, the graph of this function has infinitely many vertical asymptotes. These asymptotes occur at $x = n\pi$, where n is an integer. ■

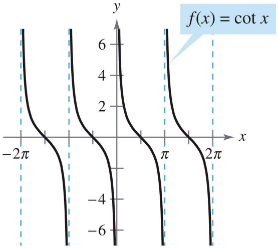


Figure 39: $f(x) = \cot x$ has vertical asymptotes at $x = n\pi$, $n \in \mathbb{Z}$.

- Theorem 1.14 requires that the value of the numerator at $x = c$ be nonzero. If both the numerator and the denominator are 0 at $x = c$, you obtain the indeterminate form $0/0$, and you cannot determine the limit behavior at $x = c$ without further investigation, as illustrated in Example 3.

Example 3 (A rational function with common factors)

Determine all vertical asymptotes of the graph of

$$h(x) = \frac{x^2 + 2x - 8}{x^2 - 4}.$$

- Begin by simplifying the expression, as shown

$$h(x) = \frac{x^2 + 2x - 8}{x^2 - 4} = \frac{(x+4)(x-2)}{(x+2)(x-2)} = \frac{x+4}{x+2}, \quad x \neq 2.$$

- At all x-values other than $x = 2$, the graph of h coincides with the graph of $k(x) = (x+4)/(x+2)$.

- So, you can apply Theorem 1.14 to k to conclude that there is a vertical asymptote at $x = -2$, as shown in Figure 40.
- From the graph, you can see that

$$\lim_{x \to -2^-} \frac{x^2 + 2x - 8}{x^2 - 4} = -\infty \quad \text{and} \quad \lim_{x \to -2^-} \frac{x^2 + 2x - 8}{x^2 - 4} = \infty. \qquad \blacksquare$$

- Note that $x = 2$ is not a vertical asymptote.

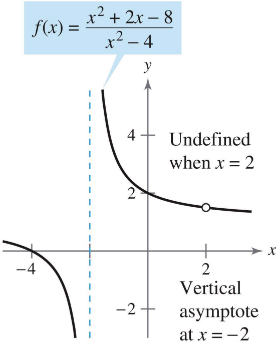


Figure 40: A rational function with common factors.

Example 4 (Determining infinite limits)

Find each limit

$$\lim_{x\to 1^-} \frac{x^2-3x}{x-1} \quad \text{and} \quad \lim_{x\to 1^+} \frac{x^2-3x}{x-1}.$$

$$\begin{aligned} \lim_{x\to 1^-} \frac{x^2-3x}{x-1} &= \infty \\ \lim_{x\to 1^+} \frac{x^2-3x}{x-1} &= -\infty \end{aligned}$$

■

Theorem 1.15 (Properties of infinite limits)

Let c and L be real numbers and let f and g be functions such that

$$\lim_{x \to c} f(x) = \infty \quad \textit{and} \quad \lim_{x \to c} g(x) = L.$$

1. *Sum or difference:* $\lim_{x \to c} [f(x) \pm g(x)] = \infty$
2. *Product:*

$$\lim_{x \to c} [f(x)g(x)] = \infty, \quad L > 0$$
$$\lim_{x \to c} [f(x)g(x)] = -\infty, \quad L < 0$$

3. *Quotient:* $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$.

1. To show that the limit of $f(x)+g(x)$ is infinite, choose $M>0$. You then need to find $\delta>0$ such that

$$[f(x)+g(x)]>M$$

whenever $0<|x-c|<\delta$.

- For simplicity's sake, you can assume L is positive. Let $M_1=M+1$. Because the limit of $f(x)$ is infinite, there exists δ_1 such that $f(x)>M_1$ whenever $0<|x-c|<\delta_1$.
- Also, because the limit of $g(x)$ is L, there exists δ_2 such that $|g(x)-L|<1$ whenever $0<|x-c|<\delta_2$.
- By letting δ be the smaller of δ_1 and δ_2, you can conclude that $0<|x-c|<\delta$ implies $f(x)>M+1$ and $|g(x)-L|<1$.
- The second of these two inequalities implies that $g(x)>L-1$, and, adding this to the first inequality, you can write

$$f(x)+g(x)>(M+1)+(L-1)=M+L>M.$$

- So, you can conclude that

$$\lim_{x\to c}[f(x)+g(x)]=\infty.$$

Example 5 (Determining limits)

a. Because $\lim_{x\to 0} 1 = 1$ and $\lim_{x\to 0} \frac{1}{x^2} = \infty$, you can write

$$\lim_{x\to 0}\left(1+\frac{1}{x^2}\right)=\infty. \qquad \textit{Property}\,1,\ \textit{Theorem}\ 1.15$$

b. Because $\lim_{x\to 1^-}(x^2+1) = 2$ and $\lim_{x\to 1^-}(\cot \pi x) = -\infty$, you can write

$$\lim_{x\to 1^-}\frac{x^2+1}{\cot \pi x}=0. \qquad \textit{Property}\,3,\ \textit{Theorem}\ 1.15$$

c. Because $\lim_{x\to 0^+} 3 = 3$ and $\lim_{x\to 0^+} \cot x = \infty$, you can write

$$\lim_{x\to 0^+} 3\cot x=\infty. \qquad \textit{Property}\,2,\ \textit{Theorem}\ 1.15$$

d. Because $\lim_{x\to 0^-} x^2 = 0$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$, you can write

$$\lim_{x\to 0^-}\left(x^2+\frac{1}{x}\right)=-\infty. \qquad \textit{Property}\,1,\ \textit{Theorem}\ 1.15 \qquad \blacksquare$$