1. (16%) Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out)

(a) $\lim_{x\to(-\frac{1}{2})} \frac{6x^2+x-1}{4x^2-4x-3}$

(b) $\lim_{x\to\frac{3\pi}{2}} \frac{cosx}{1+sinx}$

(c) $\lim_{x\to-\infty} \sqrt{x^2-2x}+x$

(d) $\lim_{x\to\infty} \sqrt{2x^2+1}\sin\frac{1}{x}$

Ans:

(a) $\lim_{x\to(-\frac{1}{2})} \frac{6x^2+x-1}{4x^2-4x-3} = \lim_{x\to(-\frac{1}{2})} \frac{(3x-1)(2x+1)}{(2x-3)(2x+1)} = \lim_{x\to(-\frac{1}{2})} \frac{(3x-1)}{(2x-3)} = \frac{5}{8}$

(b) $\lim_{x\to\frac{3\pi}{2}} \frac{cosx}{1+sinx} = \lim_{x\to\frac{3\pi}{2}} \frac{cosx(1-sinx)}{(1+sinx)(1-sinx)} = \lim_{x\to\frac{3\pi}{2}} \frac{cosx(1-sinx)}{cos^2x}$

$\lim_{x\to\frac{3\pi}{2}^+} \frac{1-sinx}{cosx} = \infty \quad \lim_{x\to\frac{3\pi}{2}^-} \frac{1-sinx}{cosx} = -\infty \to$ limit does not exit

(c) $\lim_{x\to-\infty} \sqrt{x^2-2x}+x = \lim_{x\to-\infty} \frac{(\sqrt{x^2-2x}+x)(\sqrt{x^2-2x}-x)}{(\sqrt{x^2-2x}-x)} = \lim_{x\to-\infty} \frac{-2x}{(\sqrt{x^2-2x}-x)} =$

$\lim_{x\to-\infty} \frac{-2}{(-\sqrt{1-\frac{2}{x^2}}-1)} = 1$

(d) $\lim_{x\to\infty} \sqrt{2x^2+1}\sin x\frac{1}{x} = \lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{x} x\sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{x} \lim_{t\to0^+} \frac{\sin t}{t} = \lim_{x\to\infty} \sqrt{2+\frac{1}{x^2}} = \sqrt{2}$

2. (9%)

Let $f(x) = \begin{cases} x^{\frac{4}{3}}\cos\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

(a) Is f(x) continuous at x = 0?

(b) Compute $f'(x)$ for $x \neq 0$ and $f'(0)$.

Ans:

(a)

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$

$$-x^{\frac{4}{3}} \leq x^{\frac{4}{3}}\cos\left(\frac{1}{x}\right) \leq x^{\frac{4}{3}}$$

By squeeze theorem, we have

$$\lim_{x\to0} x^{\frac{4}{3}}\cos\left(\frac{1}{x}\right) = 0$$

Therefore, it is continuous at $x = 0$.

(b) $f'(0) = \lim_{x\to 0} \frac{x^{\frac{4}{3}}\cos\left(\frac{1}{x}\right) - f(0)}{x-0} = \lim_{x\to 0} \frac{x^{\frac{4}{3}}\cos\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0} x^{\frac{1}{3}}\cos\left(\frac{1}{x}\right) = 0$ (Since

$-1 \le \cos\left(\frac{1}{x}\right) \le 1$, again by squeeze theorem)

$$f'(x) = \frac{4}{3}x^{\frac{1}{3}}\cos\left(\frac{1}{x}\right) - x^{\frac{4}{3}}\sin\left(\frac{1}{x}\right)(-x^{-2}) = \frac{4}{3}x^{\frac{1}{3}}\cos\left(\frac{1}{x}\right) + x^{\frac{-2}{3}}\sin\left(\frac{1}{x}\right)$$

3. (6%) Proof that $f(x) = x^7 + x + \frac{1}{2}$ has exactly one real root. (Hint: use the mean value theorem)

Ans:

$f(0) = \frac{1}{2}, f(-1)$

$= -\frac{3}{2}$, by the intermediate value theorem it has at least one real root between -1 and 0.

Assume the real root is a and there is a second real root b. Then by the mean value theorem, there is a c such that $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$. However, $f'(x) = 7x^6 + 1 > 0$. Contradict, therefore, there is only one real root.

4. (16%) Remember that you can solve the derivative using the definition or the differentiation rule for the following question.

(a) Find the derivative of $f(x) = \sqrt[3]{x}\left(\sqrt{x} + 3\right)$

(b) Let $f(x) = x^5 - 2x^4 + 3x + \pi$, find $f'''(x)$

(c) Let $g(x) = \tan(x^2)$, find $g'(0)$

(d) Find the following limit. $\lim_{x\to 1} \frac{\frac{x}{\sqrt{x^2+1}} - \frac{1}{\sqrt{2}}}{x-1}$

Ans:

(a) $f(x) = x^{\frac{5}{6}} + 3x^{\frac{1}{3}} \to f'(x) = \frac{5}{6}x^{\frac{-1}{6}} + x^{\frac{-2}{3}}$

(b) $f'(x) = 5x^4 - 8x^3 + 3 \to f''(x) = 20x^3 - 24x^2 \to f''(x) = 60x^2 - 48x$

(c) $g'(x) = 2xsec^2(x^2) \to g'(0) = 0$

(d) Let $f(x) = \frac{x}{\sqrt{x^2+1}}$, then $\lim_{x\to 1} \frac{\frac{x}{\sqrt{x^2+1}} - \frac{1}{\sqrt{2}}}{x-1} = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = f'(1)$.

$$f'(x) = \frac{\sqrt{x^2+1} - x(x^2+1)^{\frac{-1}{2}}}{(x^2+1)}$$

Therefore, $\lim_{x\to 1}\frac{\frac{x}{\sqrt{x^2+1}}-\frac{1}{\sqrt{2}}}{x-1}=f'(1)=\frac{1}{2}\times\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)=\frac{\sqrt{2}}{4}$

5. (8%) Given the graph $2x^2y-y^3+1=x+y$.
   (a) Express y' in terms of x and y
   (b) Find the points on the curve with $y=1$ and the tangent lines at these points

   Ans:

   (a) $\frac{d}{dx}(2x^2y-y^3+1)=\frac{d}{dx}(x+y)$

   $$4\mathrm{xy}+2x^2\frac{dy}{dx}-3y^2\frac{dy}{dx}=1+\frac{dy}{dx}$$

   $$(2x^2-3y^2-1)\frac{dy}{dx}=1-4xy$$

   $$\frac{dy}{dx}=\frac{1-4xy}{(2x^2-3y^2-1)}$$

   (b) $2x^2 1-1+1=x+1\rightarrow(2x+1)(x-1)=0\rightarrow \mathrm{x}=\frac{-1}{2},1$

   Slopes are $\frac{-6}{7},\frac{3}{2}\rightarrow\frac{-6}{7}\left(\mathrm{x}+\frac{1}{2}\right)=(y-1),\frac{3}{2}(x-1)=(y-1)$

6. (15%) Let $f(x)=\frac{(x+1)^2}{x^2+1}$
   (a) Find the open intervals on which f is increasing or decreasing. Indicates the extreme values.
   (b) Find the open intervals on which f is concave upward or concave downward. Indicates the points of inflection
   (c) Find all the asymptotes (Both vertical and horizontal)
   (d) Sketch the graph of $f(x)$
   (e) What is the domain and range of f(x)?

Ans:

(a) $f'(x)=\frac{-2(x+1)(x-1)}{(x^2+1)^2}$. The critical numbers are ± 1. f is increasing on $(-1,1)$ since $f'(x)>0$, f is decreasing on $(-\infty,-1)$ and $(1,\infty)$ since $f'(x)<0$. Local (global) maxima is (1,2) and local (global) minima is $(-1,0)$.

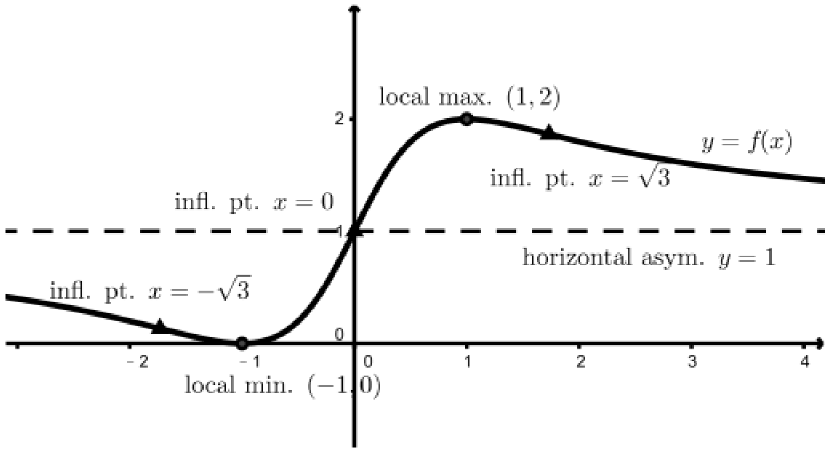
(b) $f''(x)=\frac{4x(x^2-3)}{(x^2+1)^3}$. The possible points of inflection are

$(0,1),(-\sqrt{3},\frac{(-\sqrt{3}+1)^2}{4}),(\sqrt{3},\frac{(\sqrt{3}+1)^2}{4})$. f is concave downward on $(-\infty,-\sqrt{3})$ and

$(0,\sqrt{3})$ since $f''(x) < 0$, f is concave upward on $(-\sqrt{3},0)$ and $(\sqrt{3},\infty)$ since $f''(x) > 0$. Points of inflection are $(0,1), (-\sqrt{3},\frac{(-\sqrt{3}+1)^2}{4}), (\sqrt{3},\frac{(\sqrt{3}+1)^2}{4})$.

(c) $y = 1$ is the horizontal asymptote. since $\lim_{x\to\pm\infty} f(x) = 1$. There is no vertical asymptote since $x^2 + 1 \neq 0$.

(d)



(e) Domain is entire real line. Range is [0,2]. Since the end behavior of f approaches 1 and there are no other extrema.

7. (15%) Remember the meaning and the definition of definite integral when solving the following question
   (a) $\int 2x^5 + 6x^3 - \sqrt{5}x^2 + 3\, dx$
   (b) $\int tan^2 y + 2\ dy$
   (c) $\int_{-6}^{6} \sqrt{36 - x^2} dx$
   (d) $\lim_{n\to\infty} 2(\frac{1^5+2^5+\cdots+n^5}{n^6})$
   (e) $\int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx$

Ans:

(a) $\frac{x^6}{3} + \frac{3x^4}{2} - \frac{\sqrt{5}}{3}x^3 + 3x + C$

(b) $\int tan^2 y + 2\ dy = \int (tan^2 y + 1) + 1\, dy = \int (sec^2 y) + 1\, dy = \tan y + y + C$

(c) 18π (Since it is the area of a semi-circle with radius 6)

(d) $\lim_{n\to\infty} 2(\frac{1^5+2^5+\cdots+n^5}{n^6}) = \lim_{n\to\infty} \frac{2}{n}(\frac{1^5+2^5+\cdots+n^5}{n^5}) = \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}(\frac{i}{n})^5 = 2\int_0^1 x^5\, dx =$

$2\frac{1}{6}x^6\Big]_0^1 = \frac{1}{3}$

(e) Let $u = 1 + \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}dx$

$$\int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}dx = 2\int_2^3 u^{-2}du = -2u^{-1}\Big]_2^3 = \frac{1}{3}$$

8. (9%) Show that the function $f(x) = \int_0^{\frac{1}{x}} \frac{2}{t^2+1}dt + \int_0^x \frac{2}{t^2+1}dt$ is constant for $x > 0$ (Hint: use the fundamental theorem of calculus)

Ans:

By the fundamental theorem of calculus

$$f'(x) = \frac{2}{\left(\frac{1}{x}\right)^2 + 1}\left(\frac{-1}{x^2}\right) + \frac{2}{x^2+1} = 0$$

Since the derivative of $f(x)$ is zero, f(x) is a constant.

9. (6%) Find $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (t^2 + t^6\tan(t))\,dt$.

Ans:

Note that $t^6\tan(t)$ is an odd function and t^2 is an even function

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (t^2 + t^6\tan(t))\,dt = 2\int_0^{\frac{\pi}{4}} (t^2)\,dt + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (t^2 + t^6\tan(t))\,dt = \left[\frac{2}{3}t^3\right]_0^{\frac{\pi}{4}} = \frac{\pi^3}{96}$$