

Note that e is euler constant in all the following questions

1. (20%) Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out)

$$(a) \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{\ln(\frac{2n+1}{n})}{\frac{2n+1}{n}} + \frac{\ln(\frac{2n+2}{n})}{\frac{2n+2}{n}} + \dots + \frac{\ln(\frac{2n+n}{n})}{\frac{2n+n}{n}} \right]$$

$$(b) \lim_{x \rightarrow \infty} x \tan\left(\frac{1}{x}\right)$$

$$(c) \lim_{x \rightarrow 0^+} \frac{\int_0^x \sin(t^2) dt}{x \sin(x^2)}$$

$$(d) \lim_{x \rightarrow 1^+} \frac{3}{\ln x} - \frac{2}{x-1}$$

Ans:

$$(a) \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{\ln(\frac{2n+1}{n})}{\frac{2n+1}{n}} + \frac{\ln(\frac{2n+2}{n})}{\frac{2n+2}{n}} + \dots + \frac{\ln(\frac{2n+n}{n})}{\frac{2n+n}{n}} \right] = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{\ln(2+\frac{k}{n})}{2+\frac{k}{n}} = \int_2^3 \frac{\ln x}{x} dx =$$

$$\int_{\ln 2}^{\ln 3} u du \quad (\text{let } u = \ln x, \quad du = \frac{1}{x} dx) = \frac{1}{2} u^2 \Big|_{\ln 2}^{\ln 3} = \frac{1}{2} ((\ln 3)^2 - (\ln 2)^2)$$

$$(b) \lim_{x \rightarrow \infty} x \tan\left(\frac{1}{x}\right) = \lim_{x \rightarrow \infty} \frac{\tan(\frac{1}{x})}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{-1}{x^2} \sec^2(\frac{1}{x})}{\frac{-1}{x^2}} = \lim_{x \rightarrow \infty} \sec^2\left(\frac{1}{x}\right) = 1$$

$$(c) \lim_{x \rightarrow 0^+} \frac{\int_0^x \sin(t^2) dt}{x \sin(x^2)} = \lim_{x \rightarrow 0^+} \frac{\sin(x^2)}{\sin(x^2) + 2x^2 \cos(x^2)} \quad (\text{By the fundamental theorem of calculus}$$

$$\text{and L'Hôpital's rule}) = \lim_{x \rightarrow 0^+} \frac{1}{1 + 2 \frac{x^2}{\sin(x^2)} \cos(x^2)} = \frac{1}{3}$$

$$(d) \lim_{x \rightarrow 1^+} \frac{3}{\ln x} - \frac{2}{x-1} = \lim_{x \rightarrow 1^+} \frac{3x-3-2 \ln x}{(\ln x)(x-1)} = \lim_{x \rightarrow 1^+} \frac{3-\frac{2}{x}}{\frac{x-1}{x} + \ln x} = \infty$$

2. (6%) (a) Show that $f(x) = 6 - x^3$ has an inverse function (Note that you should show that it is one to one)

$$(b) \text{Find } (f^{-1})'(7)$$

Ans:

- (a) Since $f'(x) = -3x^2$ is monotonic decreasing on $(-\infty, \infty)$. Therefore, f has an inverse.

$$(b) f^{-1}(7) = -1 \quad (\text{Since } f(-1) = 7)$$

$$(f^{-1})'(7) = \frac{1}{(f)'(f^{-1}(7))} = \frac{1}{-3(-1)^2} = \frac{-1}{3}$$

3. (6%) Given the function $f(x) = x - e \ln x$, prove that if $a > e$, then $f(a) > 0$.

(Hint: use the mean value theorem in the interval (e, a))

Ans:

Since $f(x)$ is continuous on $[e, a]$ and differentiable on (e, a) . By the mean value theorem, there exist at least one number c in (e, a) such that

$$f'(c) = 1 - \frac{e}{c} = \frac{f(a) - f(e)}{a - e} = \frac{f(a)}{a - e}$$

Since $c > e$, $\frac{e}{c} < 1$, therefore, $\frac{f(a)}{a - e} > 0$.

If $a > e$, $a - e > 0$, then $f(a) > 0$.

4. (15%) Evaluate the following integral.

(a) $\int_0^1 \frac{1}{x + \sqrt{x}} dx$

(b) $\int_0^1 \frac{x^2 + 4x}{x^3 + 6x^2 + 5} dx$

(c) $\int \frac{e^x}{e^{2x} + 2 \times e^x + 2} dx$

Ans:

(a) Let $u = \sqrt{x}$, $dx = 2u du$

$$\begin{aligned} \int_0^1 \frac{1}{x + \sqrt{x}} dx &= \lim_{b \rightarrow 0^+} \int_b^1 \frac{1}{x + \sqrt{x}} = \lim_{b \rightarrow 0^+} \int_{\sqrt{b}}^1 \frac{2}{u + 1} du = \lim_{b \rightarrow 0^+} 2 \ln |u + 1| \Big|_{\sqrt{b}}^1 \\ &= 2 \ln 2 \end{aligned}$$

(b) $u = x^3 + 6x^2 + 5, du = 3(x^2 + 4x) dx$

$$\int_0^1 \frac{x^2 + 4x}{x^3 + 6x^2 + 5} dx = \frac{1}{3} \int_5^{12} \frac{1}{u} du = \frac{1}{3} \ln |u| \Big|_5^{12} = \frac{1}{3} (\ln 12 - \ln 5)$$

(c) $u = e^x, du = e^x dx$

$$\begin{aligned} \int \frac{e^x}{e^{2x} + 2 \times e^x + 2} dx &= \int \frac{1}{u^2 + 2u + 2} du = \int \frac{1}{(u + 1)^2 + 1} du = \arctan(u + 1) \\ &+ C = \arctan(e^x + 1) + C \end{aligned}$$

5. (15%) Evaluate the following expression.

(a) Given $f(x) = x^{\sin x}$ find $f'(x)$

(b) $\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$

(c) $\int 4 \arccos(x) dx$

Ans:

(a) $\ln f(x) = \sin x \ln x$

$$\frac{f'(x)}{f(x)} = \cos x \ln x + \sin x \frac{1}{x} \rightarrow f'(x) = x^{\sin x} (\cos x \ln x + \frac{\sin x}{x})$$

(b) $\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx = \int \frac{\frac{1}{2} \sin(2x)}{(\frac{1-\cos 2x}{2})^2 + (\frac{1+\cos 2x}{2})^2} dx = \int \frac{\sin(2x)}{1+\cos^2(2x)} dx$

Let $u = \cos 2x, du = -2 \sin 2x dx$

$$= \frac{-1}{2} \int \frac{1}{1+u^2} du = \frac{-1}{2} \arctan(\cos(2x)) + C$$

(c) Let $u = \arccos(x), dv = dx, du = \frac{-1}{\sqrt{1-x^2}}, v = x$

$$\begin{aligned} \int 4 \arccos(x) dx &= 4 \left(x \arccos(x) + \int \frac{x}{\sqrt{1-x^2}} dx \right) \\ &= 4 \left(x \arccos(x) - \sqrt{1-x^2} \right) + C \end{aligned}$$

6. (20%) Evaluate the following integral. (If the integral diverges, you should point it out)

(a) $\int \frac{2x^2+3x+4}{x^3+3x^2+3x+1} dx$

(b) $\int \frac{\sec^2 x}{\tan^2 x + 5 \tan x + 6} dx$

(c) $\int_0^\infty x^3 e^{-x^2} dx$

(d) $\int_0^1 \frac{1}{\sqrt[3]{x^4}} dx$

Ans:

(a) $\int \frac{2x^2+3x+4}{x^3+3x^2+3x+1} dx = \int \frac{2x^2+3x+4}{(x+1)^3} dx = \int \frac{2}{x+1} - \frac{1}{(x+1)^2} + \frac{3}{(x+1)^3} dx = 2 \ln |x+1| +$

$$\frac{1}{x+1} - \frac{3}{2(x+1)^2} + C$$

(b)

Let $u = \tan x, du = \sec^2 x dx$

$$\begin{aligned} \int \frac{\sec^2 x}{\tan^2 x + 5 \tan x + 6} dx &= \int \frac{1}{u^2 + 5u + 6} du = \int \frac{-1}{u+3} du + \int \frac{1}{u+2} du \\ &= -\ln |u+3| + \ln |u+2| + C = \ln \left| \frac{\tan x + 2}{\tan x + 3} \right| + C \end{aligned}$$

(c) Let $u = x^2$, $du = 2x dx$

$$\begin{aligned}\int x^3 e^{-x^2} dx &= \frac{1}{2} \int u e^{-u} du = \frac{-1}{2} (u e^{-u}) + \frac{1}{2} \int e^{-u} du \\ \int_0^\infty x^3 e^{-x^2} dx &= \lim_{b \rightarrow \infty} \left. \frac{-1}{2} (u e^{-u}) \right|_0^b + \frac{1}{2} \int_0^b e^{-u} du \\ &= \frac{-1}{2} \lim_{b \rightarrow \infty} \frac{b}{e^b} - \frac{1}{2} \lim_{b \rightarrow \infty} e^{-u} \Big|_0^b = \frac{1}{2}\end{aligned}$$

(d)

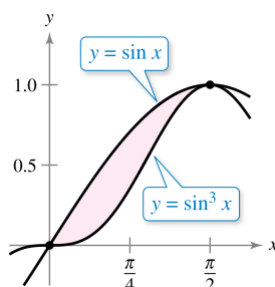
If $p = 1$, $\int_0^1 \frac{1}{x^p} dx = \int_0^1 \frac{1}{x} dx = \lim_{a \rightarrow 0^+} \ln x \Big|_a^1 = \infty$ diverges

If $p \neq 1$, $\int_0^1 \frac{1}{x^p} dx = \lim_{a \rightarrow 0^+} \left. \frac{x^{1-p}}{1-p} \right|_a^1 = \lim_{a \rightarrow 0^+} \left(\frac{1}{1-p} - \frac{a^{1-p}}{1-p} \right) = \frac{1}{1-p} - \frac{1}{1-p} \lim_{a \rightarrow 0^+} \frac{1}{a^{p-1}}$

Which converges to $\frac{1}{1-p}$ if $p - 1 < 0$

Therefore, $\int_0^1 \frac{1}{\sqrt[3]{x^4}} dx$ diverges

7. (6%) Find the following area of the region bounded by $y = \sin(x)$ and $y = \sin^3(x)$



Ans:

Let $u = \cos x$, $du = -\sin x dx$

$$\int \sin^3 x dx = \int (1 - \cos^2 x) \sin x dx = \int u^2 - 1 du = \frac{1}{3} u^3 - u + C$$

$$A = \int_0^{\pi/2} (\sin x - \sin^3 x) dx = -\cos x \Big|_0^{\pi/2} - \left(\frac{1}{3} \cos^3 x - \cos x \right) \Big|_0^{\pi/2} = \frac{1}{3}$$

8. (6%) Find the arc length of the graph $y = \ln(x)$ over the interval $[1, 5]$

Ans:

$$s = \int_1^5 \sqrt{1 + y'^2} dx = \int_1^5 \frac{\sqrt{x^2 + 1}}{x} dx$$

Let $x = \tan\theta$, $dx = \sec^2\theta d\theta$

$$\int \frac{\sqrt{x^2 + 1}}{x} dx = \int \frac{\sec\theta}{\tan\theta} \sec^2\theta d\theta = \int \frac{\sec\theta}{\tan\theta} (1 + \tan^2\theta) d\theta = -\ln|\csc\theta + \cot\theta|$$

$$+ \sec\theta + C = -\ln\left|\frac{\sqrt{x^2 + 1}}{x} + \frac{1}{x}\right| + \sqrt{x^2 + 1} + C$$

$$\begin{aligned} s &= \int_1^5 \sqrt{1 + y'^2} dx = \int_1^5 \frac{\sqrt{x^2 + 1}}{x} dx = \left[-\ln\left|\frac{\sqrt{x^2 + 1}}{x} + \frac{1}{x}\right| + \sqrt{x^2 + 1} \right]_1^5 \\ &= \ln\left(\frac{\sqrt{26} - 1}{5(\sqrt{2} - 1)}\right) + \sqrt{26} - \sqrt{2} \end{aligned}$$

9. (6%) Find the area of the surface generated by revolving the graph of $f(x) = 9 - x^2$ on the interval $[0, 3]$ about the line y-axis.

Ans:

$$S = 2\pi \int_0^3 x \sqrt{1 + f'(x)^2} dx = 2\pi \int_0^3 x \sqrt{1 + 4x^2} dx$$

Let $u = 1 + 4x^2$, $du = 8x dx$

$$S = \frac{\pi}{4} \int_1^{37} \sqrt{u} du = \frac{\pi}{6} u^{\frac{3}{2}} \Big|_1^{37} = \frac{\pi}{6} (37^{\frac{3}{2}} - 1)$$